MP467: Cosmology

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Contents

1 Overview

1.1 Newtonian Gravity

The acceleration due to gravity at the surface of a spherical mass M of radius R is

$$\mathbf{g} = -\frac{GM}{R^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit radial vector and G is Newton's universal constant of gravitation. The gravitational force on a small mass m is thus

$$\mathbf{F} = m\mathbf{g} = -\frac{GMm}{R^2}\hat{\mathbf{r}} = -\nabla U$$

where $U = -\frac{GMm}{R}$ is the potential energy of m in the gravitational field of M. The gravitational potential energy per unit mass is $V = -\frac{GM}{R}$ and $\mathbf{g} = -\nabla V$.

For a small mass in a circular orbit of radius r, with period $T = \frac{2\pi}{\omega}$, around a much larger mass M the centrifugal acceleration $r\omega^2$ is balanced by the acceleration due to gravity

$$\omega^2 r = \frac{GM}{r^2} \quad \Rightarrow \quad \omega^2 = \frac{GM}{r^3}$$

giving Kepler's third law, that the square of the period is proportional to the cube of the distance

$$T^2 = \left(\frac{2\pi}{\omega}\right)^2 = 4\pi^2\frac{r^3}{GM}.$$

Kepler's third law can easily be deduced by dimensional analysis: Newton's universal constant of gravitation has dimensions $[G] = length^3\ mass^{-1}\ time^{-2}$. For a given mass the $length^3\ time^{-2}$ gives Kepler's third law.

1.2 Why is the moon round?

Consider a spherical rocky body with mass M and radius R. The gravitational potential energy per unit mass on the surface is $V = -\frac{GM}{R}$. Rocks (e.g. SiO_2 with ${}^{28}_{14}Si$ and ${}^{16}_{8}O$) have a molecular mass $\approx 60m_P$. SiO_2 melts at $1880K \Leftrightarrow$ the chemical binding energy is $\approx k_BT = 2.6 \times 10^{-20}J = 0.16eV = E_B$ per molecule. Gravitational energy of a silica molecule on the surface of M is

$$E_{Grav} = m_{SiO_2}V.$$

Rocks will disintegrate if

$$|E_{Grav}| > E_B \tag{1.1}$$

$$E_{Grav} = -60 m_P \frac{GM}{R}.$$

For simplicity assume M has constant density

$$\Rightarrow M = \frac{4\pi}{3}\rho R^3 \Rightarrow R = \left(\frac{3M}{4\rho\pi}\right)^{1/3}$$

$$E_{Grav} = -60 m_P G M^{2/3}\left(\frac{4\rho\pi}{3}\right)^{1/3}.$$

Rocks disintegrate if

$$M^{2/3}(60 m_p G)\left(\frac{4\pi\rho}{3}\right)^{1/3} > E_B$$

$$M > \left(\frac{3}{4\pi\rho}\right)^{1/2}\left(\frac{E_B}{60 m_p G}\right)^{3/2} := M_P \tag{1.2}$$

If $M > M_P$ then we expect M will be roughly spherical (a planet or large moon), if $M < M_P$ then M will not even be approximately spherical it will be an amorphous shape. For example

$$\rho_\oplus = 5500 kg/m^3$$

$$\rho_{moon} = 3400 kg/m^3$$

$$\rho_{SiO_2} = 2800 kg/m^3$$

$$\text{let} \quad \rho = 2800 \times \alpha \times kg/m^3 \quad 1 < \alpha < 2$$

$$M_P = \frac{1}{\sqrt{\alpha}} \times (2.2. \times 10^{21}) kg$$

$$R_P = \left(\frac{3M_P}{4\pi\rho}\right)^{1/3} = \frac{570 km}{\sqrt{\alpha}}$$

$$M_{moon} = 7 \times 10^{22} kg >> M_P.$$

The largest asteroid, Ceres, has $R = 500km$ and it is roughly spherical. Small asteroids are not even approximately spherical.

1.3 How high is a mountain?

Consider a mountain of mass $M_m = N m_{SiO_2}$

Gravitational energy stored in the mountain

$$E_{Grav} \approx M_m g h.$$

Chemical Binding energy of the mountain is NE_B Mountain is stable if

$$\begin{aligned}
E_{Grav} &< NE_B \\
60m_P gh &< E_B \\
60m_P h\frac{GM}{R^2} &< E_B \\
\frac{h}{R} &< \frac{E_B R}{60m_p GM} = \frac{E_B}{60Gm_P M^{2/3}}\left(\frac{3}{4\pi\rho}\right)^{1/3} \\
&= \left(\frac{M_P}{M}\right)^{2/3} = \left(\frac{R_p}{R}\right)^2
\end{aligned}$$

e.g. for the earth with $R_\oplus = 6400km$ and $\alpha = 2$ we get

$$\frac{h}{R} = \frac{1}{\alpha}\left(\frac{570}{6400}\right)^2 = 4\times 10^{-3}$$

and

$$h = 25km.$$

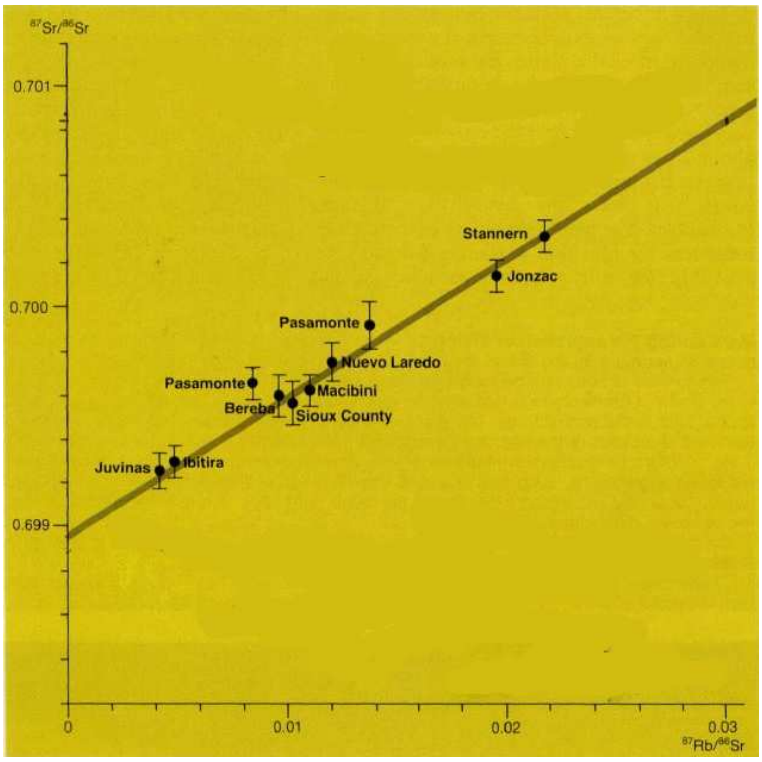
Mount Everest $\approx 10km$ This analysis ignores weathering, *i.e.* wind and freeze/thaw action, as well as tectonic events such as earthquakes. Mountains on the Earth are worn down over the millennia by weathering and earthquakes but the Himalaya, which were formed by plate tectonics when the Indian plate crashed into the Asian plate 50 million years ago, are young enough that weathering has not yet had a significant impact. On moons or planets with little or no atmosphere and no tectonic activity, these effects are absent.

1.4 How old is the Earth?

The age of the Solar System can be estimated by isotope dating, similar in concept to Carbon dating of ancient organic material but using different isotopes with a longer and more appropriate half-life. The element Rubidium, with atomic number 37, has an isotope of atomic weight 87, $^{87}_{37}$Rb, that decays via β-decay to Strontium 87 ($^{87}_{38}$St, which is stable with 38 protons and 49 neutrons) with a half-life of 5×10^{10} years

$$^{87}_{37}\text{Rb} \rightarrow \ ^{87}_{38}\text{St} + e^- + \bar{\nu}. \tag{1.3}$$

When meteorites from around the world are analysed for their $^{87}_{37}$Rb and $^{87}_{38}$Sr content it is found that a graph of the number of $^{87}_{38}$Sr atoms against the number of $^{87}_{37}$Rb atoms for various meteorites lies on a straight line with slope 0.0656.



$^{87}_{38}Sr$ v. $^{87}_{37}Rb$ content for various meteorites.

If the number of $^{87}_{38}$Sr and $^{87}_{37}$Rb atoms as a function of time are denoted by $S(t)$ and $R(t)$ respectively then

$$\begin{aligned} R(t) &= R(0)e^{-\alpha t} \\ S(t) &= S(0) + R(0)(1 - e^{-\alpha t}) = S(0) + R(t)(e^{\alpha t} - 1). \end{aligned}$$

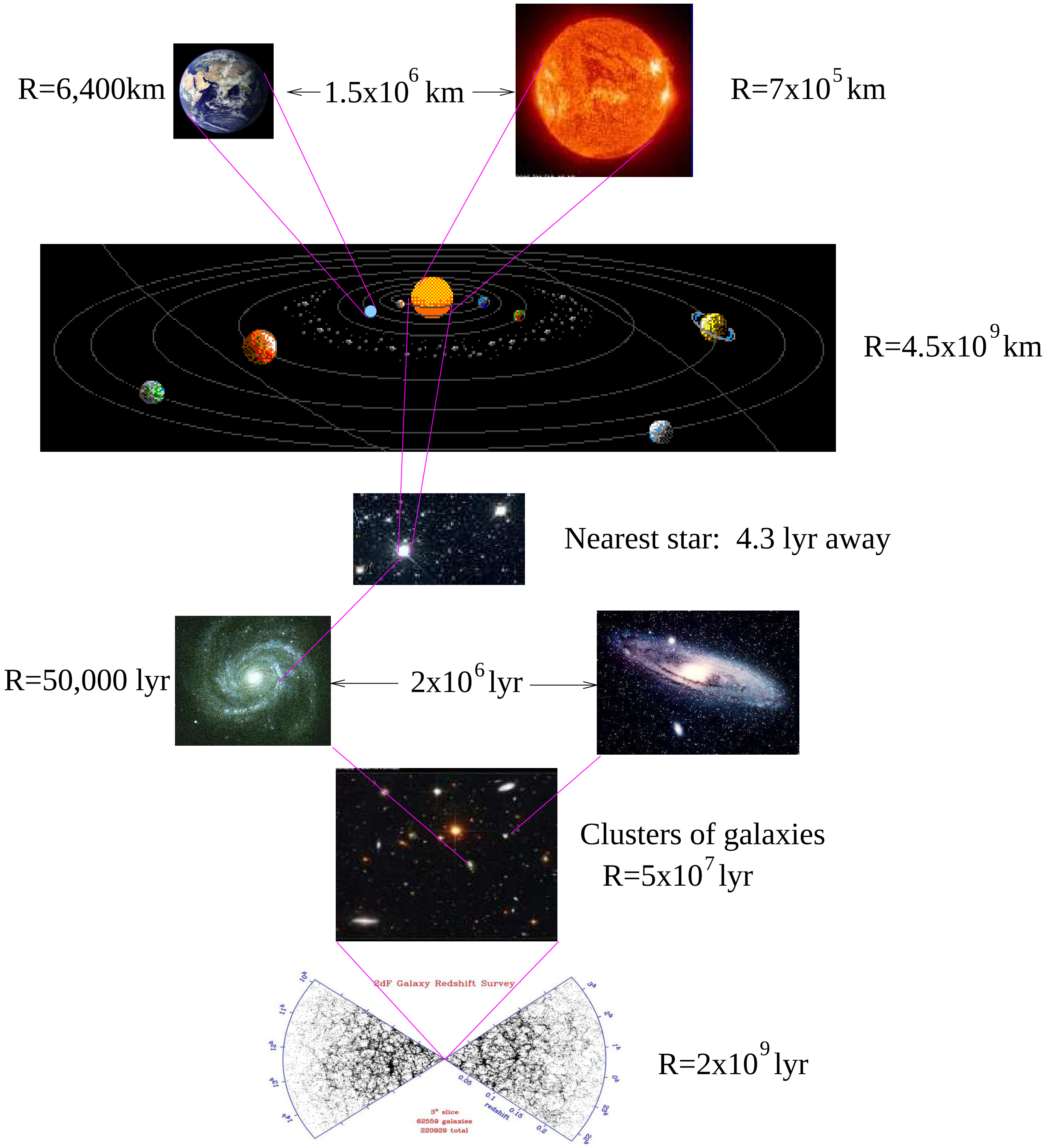
If all the meteorites are assumed to have formed about the same time, with the same initial content of $^{87}_{38}Sr$, then a plot of $S(t)$ against $R(t)$ should be indeed a straight line with slope

$$\frac{S(t) - S(0)}{R(t)} = \left(e^{\alpha t} - 1\right),$$

where $\alpha = \ln(2)/5 \times 10^{10}$ and t is measured in years. Setting

$$e^{\alpha t} - 1 = 0.0656 \qquad \Rightarrow \qquad t = \frac{\ln(1.0656)}{\alpha} = 4.6 \times 10^9 \text{yrs}.$$

The currently accepted age of the solar system is 4.56×10^9 years and the Sun has been shining, with almost constant luminosity, for this time.


R=6,400km
1.5x10^6 km
R=7x10^5 km
R=4.5x10^9 km
Nearest star: 4.3 lyr away
R=50,000 lyr
2x10^6 lyr
Clusters of galaxies
R=5x10^7 lyr
2dF Galaxy Redshift Survey
3° slice
62559 galaxies
220929 total
redshift
R=2x10^9 lyr

1.5 Distance Scales

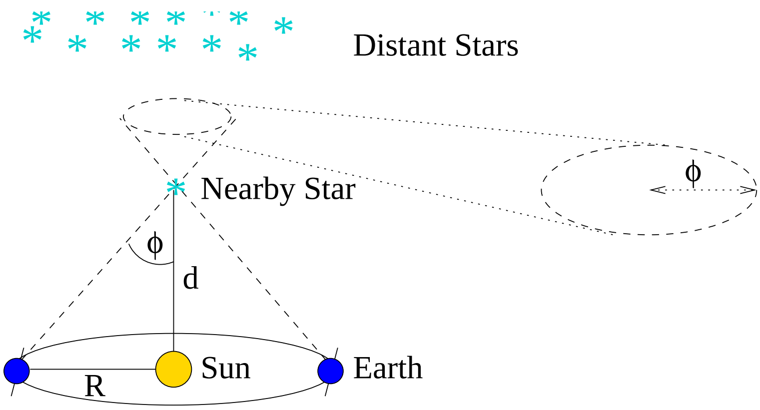
Astronomical distances are usually quoted in either *light years* or *parsecs*. A light year is the distance a beam of light travels through empty space in one year. One year is 365.25 days, or $31,557,600 secs$, so a light year is

$$1\ lyr = (2.998 \times 10^8) \times (3.156 \times 10^7) m = 9.462 \times 10^{15} m.$$

A *parsec* is defined using trigonometry with the Earth-Sun distance, $R = 1.50 \times 10^{11} m$, as a base-line. Most stars are so far away that they appear essentially stationary in the sky as the Earth moves around the Sun, changing its position by $3 \times 10^{11} m$ in 6 months. Some stars are much closer than most however and these appear to move perceptibly in the sky over a 6 month period, describing a tiny ellipse in 12 months. This is motion is called *parallax*. Simply trigonometry gives the angle ϕ as

$$R/d = \tan\phi \approx \phi \qquad \Rightarrow \qquad d \approx R/\phi,$$

where d is the distance to the star and ϕ is measured in radians.



Angles in the sky are measured in degrees divided into 60 units called *minutes of arc*, or arcminutes, and each arcminute is further divided into 60 units called *seconds of arc*, or arcseconds. The symbol $'$ is used to denote minutes of arc and $''$ to denote seconds of arc (these should not be confused with minutes and seconds of time[1]). There are thus $3600''$ in $1°$ and $1,296,000''$ in $360°$ which is 2π radians. Hence

$$1'' = 4.85 \times 10^{-6} rad.$$

A parsec (1 *psc*) is defined to be the distance at which a star would exhibit a parallax of $1''$, that is $\phi = 4.85 \times 10^{-6} rad$, so 1 *psc* corresponds to

$$d = R/(4.85 \times 10^{-6}) = \frac{1.50 \times 10^{11}}{4.85 \times 10^{-6}} m = 3.09 \times 10^{16} m = 3.26\ lyr.$$

[1]The sky appears to rotate through 360° in 24 hours which is 1° in 4 minutes, or 15 minutes of arc in 1 minute of time.

Inter-galactic distances are millions of light years, often quoted in millions of parsecs, or megaparsecs ($Mpcs$).

2 Stellar Formation and structure

2.1 Energy source

A typical star (such as the Sun) is a ball of hot plasma, mostly ionised Hydrogen. The Sun itself, for example is, 71% hydrogen, 27% Helium and only about 2% heavier elements, by mass. The surface temperature of the Sun is $T_\odot = 5800K$. It radiates energy at a rate

$$L_\odot = 4\pi R_\odot^2 \sigma_{SB} T_\odot^4 = 4 \times 10^{26} J/s$$

where σ_{SB} is the Stefan-Boltzmann constant. $L_\odot$ is called the *luminosity.* Using

$$E = mc^2 \Rightarrow 4 \times 10^{26} J/s \Leftrightarrow 4.4 \times 10^9 kg/s = -\dot{M}.$$

The mass of the Sun is $M_\odot = 2 \times 10^{30} kg$ so in principle it could sustain this rate of mass loss for $5 \times 10^{20} s$ or $1.5 \times 10^{13} yrs$. However no energy production mechanism is 100% efficient, there will always be residual mass, or "ash", left when the source of energy runs out. Possible energy sources are:

1. **Chemical Energy**: e.g. $H + H \to H_2 +$energy (typical chemical energy $\epsilon \approx 1eV$) H_2 has mass $\approx 3.4 \times 10^{-27} kg \approx 2000 MeV/c^2$. If each reaction produces about $1eV$ of energy then the efficiency η would be:

$$\eta = \frac{\epsilon}{m_{H+H} c^2} = \frac{1eV}{2000MeV} = 5 \times 10^{-10}.$$

   This could burn $M \times \eta = (2 \times 10^{30}) \times (5 \times 10^{-10}) kg = 10^{21} kg$ of the Sun's mass. The rest is left as ash H_2. This way the Sun could only last

$$\frac{M \times \eta}{|\dot{M}|} = \frac{10^{21}}{10^{10}} s = 10^{11}\ s \approx 3000\ years.$$

   The solar system was formed about 5×10^9 years ago and so the age of the Sun is $\tau_\odot \approx 5 \times 10^9 years$

2. **Gravitational Energy**: The gravitational energy between two point masses ($M_1 = M_2 = M$ a distance R apart is $E_{Grav} - \frac{GM^2}{R}$ (for a sphere of mass M, radius R and constant density it would be $E_{Grav} = -\frac{3}{5}\frac{GM^2}{R}$). Using solar values $\frac{GM_\odot^2}{R_\odot} = 3.8 \times 10^{41} J$. If the Sun were shining by converting gravitational potential

energy into heat by contracting from $R = \infty$ to its present size, assuming constant luminosity, this would take a time

$$\frac{|E_{Grav}|}{|\dot{E}|} = \frac{4.1 \times 10^{41} J}{4 \times 10^{26} J/s} = 10^{15} s = 3 \times 10^{7} years.$$

This is known as the **Kelvin-Helmholtz** time.

3. **Nuclear Energy**: $2p + 2n \rightarrow {}^{4}\mathrm{He} + 25MeV$

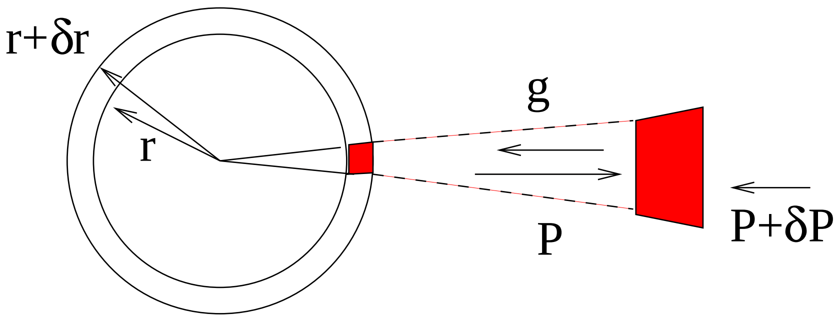
$$m_p \approx m_n \quad \Rightarrow \quad \eta = \frac{25MeV}{4m_p \times c^2} = \frac{25MeV}{4000MeV} = 6.2 \times 10^{-3}$$

$$\frac{\eta \times M_\odot}{|\dot{M}|} = \frac{(6.2 \times 10^{-3}) \times (2 \times 10^{30} kg)}{4.4 \times 10^{9} kg/s} \approx 3 \times 10^{18}\ s = 10^{11} years.$$

Only about the central 10% is actually hot enough to burn hydrogen so the expected lifetime is actually about 10^{10} years, the Sun is almost exactly at the mid-point of its life as a star.

2.2 Dynamical equilibrium

Consider a spherical mass, such as a star or a spherical gas cloud (ignoring rotation). Density, $\rho(r)$, and pressure, $P(r)$, will depend on the distance from the centre in general Consider the forces on a small volume $\delta V = \delta A \delta r$ of matter in a shell of radius r, containing mass $\delta M = \rho(r)\delta V$.



The thermal pressure on the interior face of δV is $P(r)$ (outwards) and on the exterior face is $P(r + \delta r) = P(r) + \delta P(r)$ (inwards). For a thin shell with $\delta r << 1$,

$$P(r + \delta r) \approx \frac{dP(r)}{dr}\delta r.$$

The acceleration due to gravity is $g(r) = \frac{GM(r)}{r^r}$ where

$$M(r) = 4\pi \int_0^r \rho(r) r^2 dr \tag{2.1}$$

is the mass inside the shell (gravitational forces δV due to the mass outside the shell all cancel). Newton's 2nd Law applied to the mass $\delta M = \rho(r)\delta V$ in δV gives

$$\begin{aligned}
\delta M \frac{d^2 r}{dt^2} &= -\frac{GM(r)\delta M}{r^2} + P(r)\delta A - (P(r) + \delta P(r))\delta A \\
\rho(r)\frac{d^2 r}{dt^2} &= -\frac{GM(r)\rho(r)}{r^2} - \left\{ \frac{(P(r) + \delta P(r)) - P(r)}{\delta r} \right\}.
\end{aligned}$$

Let $\delta r \to 0$

$$\rho(r)\ddot{r} = -\frac{GM(r)\rho(r)}{r^2} - \frac{dP(r)}{dr}. \tag{2.2}$$

Together with equation (2.1) in the differential form

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

this equation governs the dynamics of star and gas clouds. In summary

$$\boxed{\rho(r)\ddot{r} = -\frac{GM(r)\rho(r)}{r^2} - \frac{dP(r)}{dr}, \qquad \frac{dM}{dr} = 4\pi r^2 \rho(r)} \tag{2.3}$$

2.3 How do stars form?

Stars are formed in interstellar gas and dust clouds which start contracting under the gravitational force. A spherical interstellar gas cloud with constant density ρ_0, constant temperature T_0 and radius R, which is initially at rest will start collapsing if $\ddot{R} < 0$ and will expand if $\ddot{R} > 0$. These two possibilities are separated by the equilibrium case $\ddot{R} = 0$. To find where this boundary lies set $r = R$ and $\ddot{r} = 0$ in (2.2) and consider a cloud with uniform density and total mass M_J. Then

$$\rho_0 = \frac{3M_J}{4\pi R^3}$$

and (2.3) gives

$$\frac{GM_J}{R^2} = -\frac{1}{\rho_0}\frac{dP(R)}{dR}.$$

Assume the cloud consists of N particles with average particle mass m and obeys the ideal gas law,

$$PV = Nk_B T_0 \qquad \Rightarrow \qquad P = \frac{\rho_0}{m} k_B T_0 \qquad \Rightarrow \qquad \frac{dP}{dR} = \frac{k_B T_0}{m}\frac{d\rho_0}{dR} = -\frac{3\rho_0}{R}\frac{k_B T_0}{m}$$

$$\Rightarrow \qquad GM_J = 3R\frac{k_B T_0}{m} = 3\left(\frac{3M_J}{4\pi\rho_0}\right)^{1/3}\frac{k_B T_0}{m} \Rightarrow M_J = \frac{9}{2\sqrt{\pi}}\left(\frac{k_B T_0}{Gm}\right)^{3/2}\frac{1}{\rho_0^{1/2}}.$$

M_J is know as the **Jean's Mass** and is a critical mass for a cloud of temperature T_0 and density ρ_0. If the mass of the cloud is greater than M_J the gravitational force dominates the thermal pressure and it will collapse, if the mass is less than M_J the thermal pressure can sustain the cloud against its own gravitational contraction.

For an interstellar cloud of molecular hydrogen H_2, $m = 2m_P = 3.4 \times 10^{-27} kg$, a typical T is $\approx 100K$ and $\rho \leq 10^{-19} kg/m^3 \approx 100$ molecules$/cm^3$, we get a Jean's Mass of

$$M_J \geq 3 \times 10^{33} kg \approx 1000 M_\odot.$$

Individual stars cannot be this massive, the most massive stars known are only about $100 M_\odot$, and these are quite rare. As the cloud collapses it must break up into smaller pieces, each of which eventually becomes a star. In fact the initial stages of star formation is much more involved than this and is one of the most poorly understood stages of stellar history. For example star formation is often closely associated with shock waves in the interstellar medium which is clearly not accounted for in the above analysis. Nevertheless the conclusion that stars form in clusters rather than individually is correct.

2.4 How long does the collapse take?

For simplicity ignore P for the moment and consider a gas collapsing under free fall (a proto-star)

$$\ddot{r} = -\frac{GM(r)}{r^2}.$$

Let $r = R(t), M_0 = M(R) = const$

$$\ddot{R} = -\frac{GM_0}{R^2}.$$

Substitute $v = \frac{dR}{dt}$

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{GM_0}{R^2} \\
\Rightarrow v\frac{dv}{dt} &= -\frac{GM_0}{R^2}\frac{dR}{dt} = \frac{d}{dt}\left(\frac{GM_0}{R}\right) \\
\Rightarrow \frac{1}{2}\frac{d}{dt}(v^2) &= \frac{d}{dt}\left(\frac{GM_0}{R}\right) \\
\Rightarrow \frac{1}{2}v^2 &= \frac{GM_0}{R} + const
\end{aligned}$$

Suppose the cloud collapses from rest at $t = 0$, so $v(0) = 0$, with initial radius $R(0) = R_0$ $\Rightarrow const = -\frac{GM_0}{R_0}$ and collapses in a time t_c, so $R(t_c) = 0$. Taking $v(t)$ to be negative for $t > 0$, since the cloud is collapsing,

$$v = -\sqrt{2GM}\left(\frac{1}{R} - \frac{1}{R_0}\right)^{1/2}$$

$$\Rightarrow \frac{dt}{dR} = -\frac{1}{\sqrt{2GM_0}}\frac{\sqrt{R}}{\sqrt{1-\frac{R}{R_0}}}$$

$$\Rightarrow \int_0^{t_c} dt = -\frac{1}{\sqrt{2GM_0}}\int_{R_0}^{0}\frac{\sqrt{R}dR}{\sqrt{1-\frac{R}{R_0}}}.$$

Substituting $u = \frac{R}{R_0} = \sin^2\theta$ we get

$$\begin{aligned} t_c &= \sqrt{\frac{R_0^3}{2GM_0}}\int_0^1 \frac{\sqrt{u}du}{\sqrt{1-u}} = \\ &= \sqrt{\frac{R_0^3}{2GM_0}}\int_0^{\pi/2}\frac{2\sin^2\theta\cos\theta d\theta}{\cos\theta} \\ &= \frac{\pi}{2}\sqrt{\frac{R_0^3}{2GM_0}}. \end{aligned}$$

Assuming a constant initial density ρ_i ($M_0 = \frac{4\pi}{3}R_0^3$) we obtain

$$t_c = \sqrt{\frac{3\pi}{32G\rho_i}}$$

the free fall collapse time (zero pressure).
e.g. interstellar gas cloud $\rho = 10^{-19}kg/m^3 \Rightarrow t_c = 2\times10^{14}s = 6\times10^6 years$. For comparison: $\rho_\odot = 1.4\times10^3 kg/m^3 \Rightarrow t_c = 30min$

2.5 Virial Theorem

For a static, spherically symmetric star in equilibrium, i.e. the radius $r(t)$ of a shell is independent of t, so from (2.2) we get

$$-\frac{GM(r)\rho(r)}{r^2} = \frac{dP}{dr} \qquad (2.4)$$

The total gravitational potential energy is

$$E_{Grav} = -\int_{Star}\frac{GM(r)\rho(r)}{r}dV = -4\pi\int_0^R \frac{GM(r)\rho(r)}{r}r^2dr = 4\pi\int_0^R \frac{dP}{dr}r^3dr$$

Now assuming that $P(0)$ is finite and $P(R) = 0$

$$E_{Grav} = 4\pi \int_0^R \frac{dP}{dr} r^3 dr = 4\pi [P(r) r^3]_0^R - 12\pi \int_0^R P(r) r^2 dr =$$
$$= -12\pi \int_0^R P(r) r^2 dr$$

Define the average pressure

$$\bar{P} = \frac{\int_{Star} P dV}{V} = \frac{4\pi \int_0^R P(r) r^2 dr}{V} \tag{2.5}$$

so

$$\bar{P} = -\frac{E_{Grav}}{3V}.$$

We can also relate $\bar{P}$ to the thermal energy for an ideal monatomic gas using the Equipartition Theorem from thermodynamics which states that, for a gas in thermal equilibrium at temperature T, every dynamical degree of freedom associated with the gas particles carries energy $k_B T/2$. For a monatomic gas for example, in which the only dynamical degrees of freedom are translations in each of the three independent direction of space,

$$E_{Thermal} = \tfrac{3}{2} N k_B T.$$

Essentially the kinetic energy of each particle is

$$\epsilon_{Kinetic} = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)$$

and the equipartition theorem states that there is an energy $\frac{k_B T}{2}$ associated with motion in each direction in space

$$\frac{m}{2} v_x^2 = \frac{m}{2} v_y^2 = \frac{m}{2} v_z^2 = \frac{k_B T}{2}$$

giving

$$E_{Thermal} = N \epsilon_{Kinetic} = \frac{3}{2} N k_B T.$$

To use the equation of state for an ideal gas in modelling a star we replace the pressure P with its average value in the star, $\bar{P}$ in (2.5), and the temperature T with the average temperature $\bar{T}$, defined similarly

$$\bar{T} = \frac{\int_{Star} T dV}{V} = \frac{4\pi \int_0^R T(r) r^2 dr}{V}.$$

We can then write

$$\bar{P} V = N k_B \bar{T} = \tfrac{2}{3} E_{Thermal} = -\frac{E_{Grav}}{3}$$

For a (non-relativistic) ideal gas in thermodynamic equilibrium we obtain the Virial Theorem

$$2E_{Thermal} + E_{Grav} = 0 \tag{2.6}$$

If a star contracts slightly, in quasi-equilibrium,[1] then $\bar{T} \to \bar{T} + \Delta\bar{T}$ and $E_{Grav} \to E_{Grav} + \Delta E_{Grav}$ (we expect $\Delta E_{Grav} < 0$ for a contraction). Then the virial theorem says: $\Delta E_{Thermal} = -\frac{1}{2}\Delta E_{Grav} > 0 \Rightarrow \Delta\bar{T} > 0$, as the star contracts the temperature goes up. Half of $|E_{Grav}|$ goes into heating the star up. The other half must be radiated away (conservation of energy) as thermal radiation.

If a star expands, temperature decreases and $\Delta E_{Grav} > 0$. Half of ΔE_{Grav} comes from thermal energy, the other half must come from somewhere else! e.g. when the Sun runs out of hydrogen fuel (about 5 billion years from now), initially the core will contract and heat up as E_{Grav} is converted into $E_{Thermal}$. Eventually the temperature becomes high enough for He to burn which releases more nuclear energy which is transferred to the outer envelope which provides the extra energy necessary for the outer envelope to expand and cool - the star becomes a Red Giant. Without some source of energy other than gravitation a star in quasi-static equilibrium could never expand, contraction would be a one-way process.

2.6 Stellar Equilibrium

For a static equilibrium (normal state) $\dot{r} = 0$ we had

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{2.7}$$

The mass in a shell of thickness δr is

$$\delta M = 4\pi\rho(r)r^2\delta r$$

$$\frac{dM}{dr} = 4\pi r^2\rho(r) \tag{2.8}$$

Combining these two equations we get

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G\rho(r).} \tag{2.9}$$

We have one differential equation for two unknown functions $P(r), \rho(r)$. We need another relation between P and ρ to solve the problem and there will be more on this later.

[1] "Quasi-equilibrium" means that the state changes very slowly relative to the time scale necessary to achieve thermal equilibrium. Imagine changing the radius of a star in a series of very small steps, allowing time for thermal equilibrium to be maintained at each step.

For the moment we take a short cut by making a simplistic assumption, that the density is uniform: $\rho(r) = \rho_0 = const$ so $M(r) = \frac{4\pi}{3}\rho_0 r^3$. Then it is probably easier to solve (2.7) directly:

$$\frac{dP(r)}{dr} = -\frac{4\pi G\rho_0^2 r}{3} \qquad \Rightarrow \qquad P(r) = C - \frac{2\pi G\rho_0^2 r^2}{3}$$

where C is a constant. Assuming that the pressure vanishes at the surface of the star $P(R) = 0$ fixes $C = \frac{2\pi G\rho_0^2 R^2}{3}$ giving

$$P(r) = \frac{2\pi G\rho_0^2}{3}(R^2 - r^2)$$

and the central pressure is

$$P_c := P(0) = \frac{2\pi G\rho_0^2 R^2}{3} = \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_0^{4/3}.$$

For the Sun: $M_\odot = 2\times10^{30}kg,\ R_\odot = 7\times10^8 km$ and $\rho_\odot = 1.4\times10^3 kgm^{-3} = 1.4gcm^{-3}$ giving

$$P_c = 1.3\times10^{14} Nm^{-2}.$$

Better, more sophisticated, models give

$$P_c = 2.5\times10^{16} Nm^{-2} \quad \text{and} \quad \rho_c = 1.62\times10^5 kgm^{-3}$$

$\rho = const$ is too simplistic; better is to use the relation between pressure, density and temperature: the equation of state, $P(\rho, T)$.

2.7 Equation of State

i) **Ideal gas law**: $PV = Nk_BT$

$$P = \frac{\rho k_B T}{m}$$

where m is the average mass of the gas particles.
For a star made of ionised hydrogen, $H^+ + e^-$, $\Rightarrow m = \frac{m_p + m_e}{2} \approx \frac{m_p}{2}$

$$P = \frac{2\rho k_B T}{m_p}.$$

e.g. in our simple model ($\rho = \rho_0 = const$)

$$T_c = \frac{m_p P_c}{2k_B\rho_0} = \frac{1}{4}\left(\frac{4\pi}{3}\right)^{1/3}\frac{Gm_p}{k_B}M^{2/3}\rho_0^{1/3} = \frac{Gm_pM}{4R}.$$

For the Sun we get

$$T_{c\odot} = 6\times10^6 K$$

(more accurate models give $T_{c\odot} = 1.56 \times 10^7 K$).

Notice that

$$\begin{aligned} P_c &= \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho_0^{4/3} = \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\left(\frac{m_p P_c}{2k_B T_c}\right)^{4/3} \\ \Rightarrow \qquad P_c &= \frac{6}{\pi G^3}\left(\frac{2k_B}{m_p}\right)^4 \frac{T_c^4}{M^2} \end{aligned} \tag{2.10}$$

so, if two stars have the same central temperature, the more massive one has a *lower* central gas pressure.

The pressure is modified slightly for a more realistic chemistry. For a star made of a mixture of Hydrogen and ^{4}He, both completely ionised, let X denote the mass fraction of Hydrogen and Y the mass fraction of Helium, so $X + Y = 1$. Then if n_p is the number density of Hydrogen nuclei (protons) and n_{He} the number density Helium nuclei (each with mass $m_{He} = 4m_p$)

$$\rho = (X + Y)\rho = m_p n_p + m_{He} n_{He} = m_p n_p + 4m_p n_{He}$$

with

$$\begin{aligned} X =& \frac{m_p n_p}{\rho} \qquad \Rightarrow \qquad n_p = \frac{X\rho}{m_p} \\ Y =& 4\frac{m_p n_{He}}{\rho} \qquad \Rightarrow \qquad n_{He} = \frac{Y\rho}{4m_p}. \end{aligned}$$

The number density of gas particles is

$$n = n_e + n_p + n_{He} = 2n_p + 3n_{He} = 2\left(\frac{X\rho}{m_p}\right) + 3\left(\frac{Y\rho}{4m_p}\right) = \frac{\rho}{m_p}\left(2X + \frac{3}{4}Y\right),$$

since there is an electron for every proton and 2 electrons for every Helium nucleus. So the average mass of the gas particles is

$$m = \frac{\rho}{n} = \frac{m_p}{2X + (3/4)Y}.$$

For example the Sun has $X = 0.7$ and $Y = 0.3$ on average, but it is converting Hydrogen into Helium only in the core and has already used up half of its fuel so in the core one expects $X = 0.35$ and $Y = 0.65$ at the moment giving

$$m = \frac{m_p}{1.2},$$

rather than the value $m_p/2$ that we got for pure Hydrogen. This reduces the pressure by a factor 1.2/2=0.6. Using a central temperature $T_{c\odot} = 1.56 \times 10^7\ K$ and $\rho_c = 1.62 \times 10^5\ kgm^{-3}$ the ideal gas law gives $P_c = 2.5 \times 10^{16}\ Nm^{-2}$. As the Sun ages X in the core will decrease and Y will increase, thus reducing the pressure.

ii) **Thermal Radiation pressure**:

$$P = \frac{a}{3}T^4$$

$a = \frac{4}{c}\sigma_{SB} = 7.5 \times 10^{-16} Jm^{-3}K^{-4}$.
Using $T_{c\odot} = 1.6 \times 10^7 K$ gives

$$P_{Radiation} = (2.5 \times 10^{-16}) \times (6.6 \times 10^{28}) = 1.7 \times 10^{13}\ Nm^{-2}$$

which is much less than the gas pressure $P_c = 2.5 \times 10^{16} Nm^{-2}$.

However for a given temperature the radiation pressure is independent of the mass of the star while we have seen that the gas pressure decreases as $\sim 1/M^2$ as the mass is increased. The pressures are equal when

$$\frac{6}{\pi G^3}\left(\frac{2k_B}{m_p}\right)^4 \frac{T_c^4}{M^2} = \frac{a}{3}T_c^4 \qquad \Rightarrow \qquad M^2 = \frac{3}{a}\frac{6}{\pi G^3}\left(\frac{2k_B}{m_p}\right) = 4 \times 10^{31}\ kg = 20M_\odot.$$

For the Sun it is gas pressure that supports it, radiation pressure is negligible. For a star with $M > 2M_\odot$ radiation pressure becomes more significant. For $M > 100M_\odot$ thermal pressure dominates and the center of the star is supported purely by thermal radiation pressure!

iii) **Degenerate Fermi Gas**: Relevant for very high densities of particles with intrinsic spin of $\frac{1}{2}\hbar$. Such particles obey the Pauli exclusion principle and are called Fermions. If we try to put a lot of Fermions (e.g. electrons or neutrons) into a small box, the exclusion principle generates a pressure which resists the addition of more particles. This is called the *degeneracy pressure* because, as we shall see, it depends on the fact that quantum mechanical energy levels have finite degeneracy. It is partly responsible (together with electrostatic repulsion) for the force on the soles of your feet that stops you falling through the floor

Consider N particles confined in a cubical volume $V = a^3$. Quantum states are superpositions of standing waves, with wave-length λ. Define the *wave-number*, k, to be $k = \frac{2\pi}{\lambda}$. Quantum mechanics only allows discrete wave-vectors $\vec{k} = (m_x, m_y, m_z)\frac{2\pi}{a}$, where m_x, m_y and m_z are integers, because a whole number of wavelengths must fit in the box of width a.

The number of quantum states in a volume $d^3k = dk_x dk_y dk_z$ of wave-vector space is $dm_x dm_y dm_z = \left(\frac{a}{2\pi}\right)^3 d^3k$ at least for large integers m_x, m_y, m_z.

Now the number of quantum states in a spherical shell of thickness dk and radius k in k-space is

$$\left(\frac{a}{2\pi}\right)^3 4\pi k^2 dk = \frac{V}{(2\pi)^3}4\pi k^2 dk.$$

In quantum mechanics the de Broglie relation is $p = \frac{h}{\lambda} = \hbar k$, so the number of quantum states for particles with momentum in the range $p \to p + dp$ is

$$\frac{V}{(2\pi)^3}4\pi\left(\frac{p}{\hbar}\right)^2 d\left(\frac{p}{\hbar}\right) = \frac{4\pi V}{h^3}p^2 dp =: g(p)dp.$$

$g(p)$ is called degeneracy factor. Because of electron (or neutron) spin, there can be two states with the same p, spin up and spin down $\Rightarrow$ for electrons and neutrons

$$g(p) = \frac{8\pi V}{h^3} p^2. \tag{2.11}$$

Possible values of a particle's energy, $\epsilon(p)$ are also discrete (when confined to a box).

Let the lowest possible energy be ϵ_0. If N electrons go into the box, all possible energy levels will be filled sequentially up to some top energy $\epsilon_F = \epsilon(p_F)$. ϵ_F (the Fermi-energy) and p_F (the Fermi-momentum) depend on N and are given, when N is large, by approximating the sum over energy levels, or equivalently momenta, by an integral

$$\begin{aligned} N &= \int_0^{p_F} g(p)dp = \frac{8\pi V}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi V}{3h^3} p_F^3 \qquad \Rightarrow \\ p_F &= h \left(\frac{3}{8\pi} \frac{N}{V} \right)^{1/3} = h \left(\frac{3n}{8\pi} \right)^{1/3} \end{aligned} \tag{2.12}$$

where $n := N/V$ is the number of particles per unit volume.

We want to calculate $P(\rho, T)$, but the calculation is simplified when T is "small" (by which is meant $k_B T \ll$ energy level differences, $\Delta\epsilon$), in which case the Fermi gas is said to be *degenerate*. The internal energy is then

$$E = \int_0^{p_F} \epsilon(p) g(p) dp = \frac{8\pi}{h^3} V \int_0^{p_F} \epsilon(p) p^2 dp \tag{2.13}$$

with $\epsilon(p) = \frac{p_F^2}{2m}$ for non-relativistic momenta and $\epsilon(p) = \sqrt{m^2c^4 + p^2c^2}$ for relativistic momenta (the relativistic form is used below as the non-relativistic form can then follows by taking $c \to \infty$).

The first Law of Thermodynamics, for an adiabatic process, is[2]

$$dE = -PdV + TdS + \mu dN.$$

For $T \approx 0$ this reduces to $dE = -PdV + \mu dN$ so, using (2.13) and noting that $\left(\frac{\partial g(p)}{\partial V}\right)_N = \frac{g(p)}{V}$ from (2.11),

$$\begin{aligned} P &= -\left(\frac{\partial E}{\partial V}\right)_N = -\frac{E}{V} - \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \left(\frac{\partial p_F}{\partial V}\right)_N \\ \mu &= \left(\frac{\partial E}{\partial N}\right)_V = \frac{8\pi V}{h^3} \epsilon(p_F) p_F^2 \left(\frac{\partial p_F}{\partial N}\right)_V . \end{aligned}$$

[2] μ is the chemical potential, i.e. the energy required to add one extra particle — we shall prove momentarily that $\mu = \epsilon_F$.

Now from (2.13)

$$\left(\frac{\partial p_F}{\partial V}\right)_N = -\left(\frac{1}{3V}\right)p_F, \qquad \left(\frac{\partial p_F}{\partial N}\right)_V = \left(\frac{1}{3N}\right)p_F$$

hence

$$\begin{aligned} P &= -\frac{E}{V} + \frac{8\pi V}{h^3}\epsilon(p_F)p_F^2\frac{p_F}{3V} = -\frac{E}{V} + n\epsilon(p_F) \\ \mu &= \frac{8\pi V}{h^3}\epsilon(p_F)p_F^2\frac{p_F}{3N} = \frac{8\pi V}{3h^3 n}\epsilon(p_F)p_F^3 = \epsilon(p_F). \end{aligned}$$

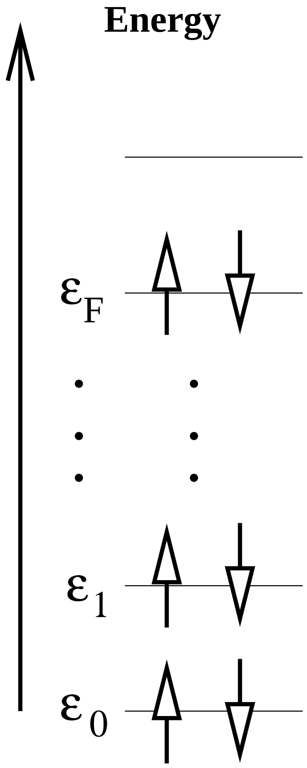
$$\Rightarrow \qquad P = \frac{\epsilon_F N - E}{V}, \qquad \mu = \epsilon_F. \tag{2.14}$$

Now

$$\begin{aligned} P &= \epsilon_F n - \frac{E}{V} = \\ &= mc^2\sqrt{1+\frac{p_F^2}{m^2c^2}}\frac{8\pi}{3}\left(\frac{p_F}{h}\right)^3 - \frac{8\pi mc^2}{h^3}\int_0^{p_F}\sqrt{1+\frac{p^2}{m^2c^2}}p^2dp \\ &= \frac{8\pi mc^2}{h^3}(mc)^3\left\{\frac{1}{3}\sqrt{1+z_F^2}z_F^3 - \int_0^{z_F}\sqrt{1+z^2}z^2dz\right\} \quad \left(z=\frac{p}{mc},\ dp = mcdz\right) \\ &= \frac{8\pi m^4c^5}{h^3}\int_0^{z_F}\frac{z^4}{3\sqrt{1+z^2}}dz \quad \left(z=\sinh\alpha,\ \sqrt{1+z^2}=\cosh\alpha,\ dz=\cosh\alpha d\alpha\right) \\ &= \frac{\pi m^4c^5}{h^3}\left\{z_F\left(\frac{2}{3}z_F^2-1\right)\sqrt{1+z_F^2} + \ln\left(z_F+\sqrt{1+z_F^2}\right)\right\} \\ &\longrightarrow \begin{cases} \frac{8\pi}{15}\frac{m^4c^5}{h^3}z_F^5, & z_F \ll 1,\ p_F \ll mc \text{ (non-relativistic)} \\ \frac{2\pi m^4c^5}{3h^3}z_F^4, & z_F \gg 1,\ p_F \gg mc \text{ (relativistic)} \end{cases} \end{aligned}$$

$$P = \begin{cases} \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m}n^{5/3} & \text{non-relativistic} \\ \left(\frac{3}{8\pi}\right)^{1/3}\frac{ch}{4}n^{4/3} & \text{relativistic.} \end{cases} \tag{2.15}$$

It has been assumed that the temperature is negligible, these expressions are valid, provided $k_BT \ll \epsilon_F$. This is the equation of state for a "‘cold"’ degenerate gas of electrons or neutrons but note that, when n is large, ϵ_F can be so large that T is negligible even for temperatures of many thousands of degrees.



Discrete energy levels, filled according to the exclusion principle.

Summary: Three different equations of state, depending on the situation:

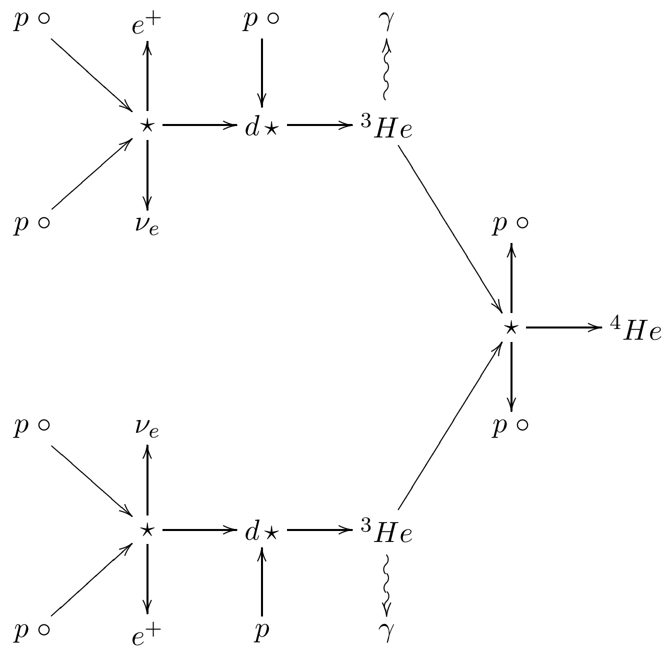
$$P(\rho, T) = \begin{cases} \frac{\rho k_B T}{m} & \text{Ideal gas pressure} \\ \frac{aT^4}{3} & \text{Radiation pressure} \\ \propto n^\beta & \beta = \begin{cases} 5/3 & \text{non-relativistic} \\ 4/3 & \text{relativistic} \end{cases} \end{cases} \qquad \text{where } n = \rho/m.$$

To solve the problem completely we need $T(r)$ - depends on energy transport and energy production.

2.8 Energy Sources in Stars

Three main nuclear reactions in stars

1. **Hydrogen burning**. $4H \rightarrow^4$He, relevant for $T \geq 10^6 K$ (the *pp*-chain). This is the main source of power in the Sun.



Net result:

$$4p \rightarrow \ ^4He + 2e^+ + 2\nu_e + 2\gamma \ (26.2MeV).$$

Each step has a characteristic time-scale:

$$\begin{array}{rl} p + p \rightarrow d + e^+ + \nu_e & \tau \approx 10^{10} yr = 3 \times 10^{17} s \\ p + d \rightarrow \ ^3He & \tau = 1s \\ ^3He + \ ^3He \rightarrow \ ^4He + 2p & \tau = 3 \times 10^5 yrs \end{array}$$

($d =^2$H is a deuterium nucleus, an isotope of Hydrogen with one neutron and one proton). The rate determining step, $pp \rightarrow d$, is so slow that it cannot be measured in the laboratory and must be calculated. Estimates range from 6×10^9 *yr* to 14×10^9 *yr*.

For the Sun, with $X = 0.35$ the mass fraction of Hydrogen in the core,

$$\rho_c = 1.6{\times}10^5 kgm^{-3} \quad \Rightarrow \quad n_p = X\frac{\rho_c}{m_p} = \frac{0.35 \times (1.6 \times 10^5 kgm^{-3})}{1.7 \times 10^{-27} kg} = 3.3{\times}10^{31} m^{-3}.$$

The rate per unit volume for the reaction $p + p \rightarrow d$ is then

$$\frac{n_p}{\tau} = \frac{3.3 \times 10^{31} m^{-3}}{3 \times 10^{17} s} = 1.1 \times 10^{14} s^{-1} m^{-3}.$$

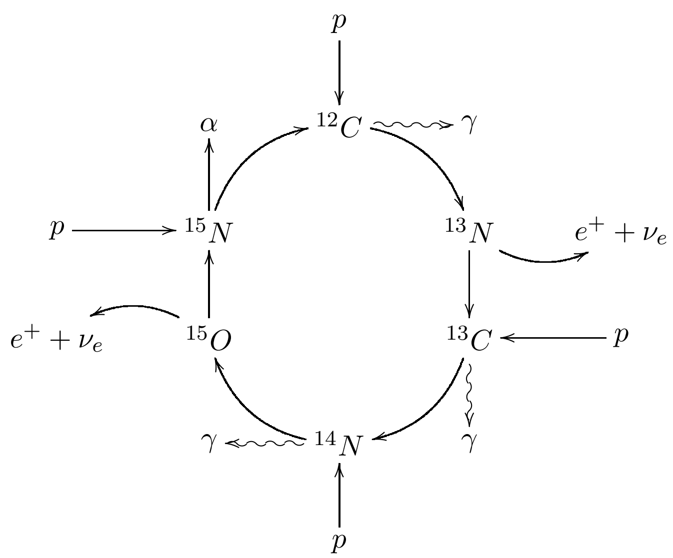
The power/unit volume requires an extra factor of 1/2 because it requires two $p + p \to d$ reactions to produce one ^{4}He nucleus, liberating 26.2 MeV:

$$
\begin{aligned}
w_{pp} &= \frac{1}{2}(1.1 \times 10^{14} s^{-1} m^{-3}) \times (26.2 MeV) = \\
&= \frac{1}{2}(1.1 \times 10^{14}) \times (26.2 \times 1.6 \times 10^{-13} J s^{-1} m^{-3}) \\
&= 231 W m^{-3}
\end{aligned}
$$

$$
\Rightarrow \qquad \frac{w_{pp}}{\rho_c} = \frac{231}{1.6 \times 10^5} = 1.4 \times 10^{-3} W kg^{-1}. \qquad (2.16)
$$

This is the dominant source of energy production in the Sun.

2. **CNO-cycle**
For $M \gg M_\odot$ central temperatures are higher and another reaction kicks in .



Net result:

$$4p \to \; ^4He + 2e^+ + 2\nu_e + 3\gamma \; (23.8 MeV).$$

Slowest step: $p + ^{14}N \to ^{15}O + \gamma$, $\tau = 5 \times 10^8 yr = 1.5 \times 10^{16} s$. But in the Sun only $\approx 0.6\%$ is ^{14}N and only 1.6% of the Sun's energy is CNO. For stars with mass a little more than the Sun the CNO cycle dominates.

3. **Helium burning**. The CNO cycle does not produce C, N or O. ^{12}C can be produced by Helium burning. In the Sun, once H is used up, ^4He can burn $(10^8 < T < 2 \times 10^8 K)$.

$$
\begin{aligned}
&^4He + ^4He \rightleftharpoons ^8Be && \text{unstable, } \tau = 2.6 \times 10^{-16} s \\
&^4He + ^8Be \rightleftharpoons ^{12}C^* && \text{unstable, } \tau = 1.8 \times 10^{-16} s \\
&\quad ^{12}C^* \to ^{12}C + \gamma \; (7.3 MeV) &&
\end{aligned}
$$

Once ^{12}C is present other reactions are possible, such as $^{4}\text{He} + ^{12}\text{C} \rightarrow ^{16}\text{O} + \gamma$. The CNO-cycle can contribute significantly, even when H is plentiful, if M is high enough. For $M > 8M_\odot$ temperatures can reach $T \sim 5 \times 10^8$, which is hot enough to burn higher nuclei up to ^{56}Fe. Nuclei higher than ^{56}Fe are destroyed in stellar interiors, not created.

The main source of energy in the Sun is the *pp*-chain, the CNO cycle and Helium burning are important in more massive stars and Helium burning will become important when the Sun runs out of Hydrogen 5 billion years from now. We saw that the *pp*-chain gave a value

$$\frac{w_c}{\rho_c} = 1.4 \times 10^{-3} W kg^{-1}. \tag{2.17}$$

If the whole Sun were burning nuclear fuel we would expect the total power production to be

$$L = M_\odot \frac{w_c}{\rho_c} = (2 \times 10^{30}) \times (1.4 \times 10^{-3}) = 2.8 \times 10^{27} J s^{-1}$$

whereas the observed luminosity is $L_\odot = 3.8 \times 10^{26} J s^{-1}$, implying that slightly more than 10% of the Sun's mass is involved in nuclear burning. This is in the central core where it is hottest. We could try to estimate the radius, R_c, of the nuclear burning central region by demanding that it contains $\approx 1/10$ of the total mass. Assuming the core has uniform density,

$$\frac{M_\odot}{10} = \frac{4\pi}{3} \rho R_c^3 \qquad \Rightarrow \qquad R_c = \left(\frac{3M_\odot}{40\pi\rho} \right)^{1/3}.$$

For example $\rho = \rho_\odot = 1.4 \times 10^3 kg\, m^{-3}$ gives $R_c = 3.2 \times 10^8 m = 0.5 R_\odot$ while $\rho = \rho_c = 1.6 \times 10^5 kg m^{-3}$ gives $R_c = 6.7 \times 10^7 m = 0.1 R_\odot$. Clearly the first is an overestimate and the second an underestimate. A more accurate value is $R_c \approx 0.3 R_\odot$.

An important consequence of the *pp*-chain nuclear reaction is neutrino production. Using

$$L_\odot = 4 \times 10^{26} J s^{-1} = \frac{4 \times 10^{26} eV s^{-1}}{1.6 \times 10^{-19}} = 2.5 \times 10^{45} eV s^{-1} = 2.5 \times 10^{39} MeV s^{-1}$$

and each *pp*-chain reaction produces $26.2 MeV$ so the number of reactions per second is $2.5 \times 10^{39}/26.2$. Each reaction produces two neutrinos so the expected rate of neutrino production is

$$\left(2 \times \frac{2.5 \times 10^{39}}{26.2} \right) s^{-1} = 1.9 \times 10^{38} s^{-1}$$

giving an expected flux at the Earth of

$$\frac{1.9 \times 10^{38} s^{-1}}{4\pi d^2_{\oplus - \odot}} = \frac{1.9 \times 10^{38}}{4\pi (150 \times 10^9 m)^2} = 6.7 \times 10^{14} m^{-2} s^{-1}.$$

Only 1/2 of this is measured and this is the famous Solar neutrino problem which has important implications for particle physics — some of the electron neutrinos in the Sun are changing into other particles, a different kind of neutrino, as they pass between the Sun and the Earth.

2.9 Energy transport

Only the central core of stars actually generates energy by nuclear reactions and this must be transported to the surface in order to explain the surface luminosity. Energy can be transported by three different mechanisms: convection, conduction or radiation. For a typical star like the Sun, radiation dominates energy transport in the core (convection dominates further out). For the most part conduction is not significant for normal stars and will not be considered here (though it can be important in the final stages of a star's life: white dwarves).

2.9.1 Radiation

Absolute luminosity = total power radiated

$$L_{\odot} = 3.8 \times 10^{26} J/s \text{ from the surface.}$$

For an object with surface area A at a temperature T radiating into empty space

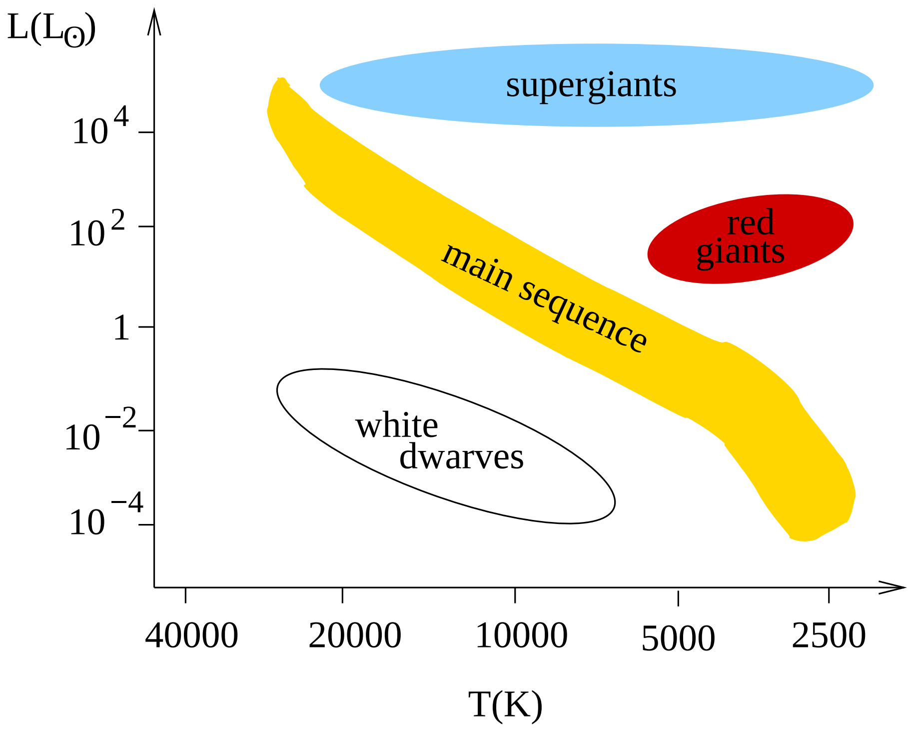
$$L = \sigma_{SB} A T^4 \tag{2.18}$$

(this is called *black body* radiation). For a spherically symmetrical object with radius R

$$L = 4\pi R^2 \sigma_{SB} T^4$$

$$\ln L = 4 \ln T + 2 \ln R + \ln(4\pi\sigma_{SB}).$$

This relation is evident if the luminosity of a large number of stars is plotted against their surface temperature on a log-log plot. Stars with similar radii fall on a straight line. Such a plot is called a *Hertzprung-Russel* diagram.



Hertzsprung-Russel diagram.

(It is conventional to plot the temperature as increasing to the left.) Most (normal) stars fall on a single line, called the *main sequence*, indicating that they all have much the same radius. There is a group of hot smaller stars falling below the main sequence, called *white dwarves.* There are also two groups of larger stars above the main sequence, indicating larger than normal radii, one slightly to the right (indicating a lower surface temperature — a group called *red giants*) and with lower luminosity than the other. Members of the higher luminosity group have very large radii are called *supergiants.*

The Hertzsprung-Russel diagram indicates the surface luminosity of stars but the energy source is only in the core, the outer layers are not hot enough to burn nuclear fuel. There must therefore be a flux of energy through any shell of matter of radius $r < R$

$$\text{Flux: } \mathcal{F}(r) = \frac{L(r)}{4\pi r^2}. \tag{2.19}$$

The flux of energy through a given sphere depends on r in general, as each nuclear burning shell is producing energy. Nuclear reactions in the core of a star produce energy at a rate per unit volume $w(r)$ and w will depend on r through ρ and T, $w(\rho, T)$ and this requires a second equation of state (e.g. $w \propto \rho T^q$).

In equilibrium conservation of energy demands that the total power radiated outward from the outer surface of a shell at radius r of thickness δr, $L(r + \delta r)$, must equal the total power going into the shell at r plus the power produced by nuclear reactions within the shell:

$$
\begin{aligned}
L(r+dr) - L(r) &= 4\pi r^2 dr w(r) \\
\Rightarrow \qquad \frac{dL}{dr} &= 4\pi r^2 w(r)
\end{aligned}
\tag{2.20}
$$

The energy flux, $\mathcal{F}(r)$, is driven by radiation pressure,

$$
P_{Rad} = \frac{a}{3} T^4 \qquad a = \frac{4\sigma_{SB}}{c}.
$$

It is reasonable to assume that $\mathcal{F} \propto \frac{dP_{Rad}}{dr}$

$$
\mathcal{F} = -\frac{c}{\rho\kappa}\frac{dP_{Rad}}{dr} \tag{2.21}
$$

where κ is called the "opacity" and has dimensions of area over mass.

$$
\mathcal{F}(r) = \frac{L(r)}{4\pi r^2} = -\frac{c}{\rho\kappa}\frac{dP_{Rad}}{dr} = -\frac{ca}{3\rho\kappa}\frac{d}{dr}(T^4)
$$

$$
\boxed{\frac{d}{dr}(T^4) = -\frac{3}{4\pi r^2}\frac{\rho\kappa}{ca}L(r).}
$$

This equation relates the temperature gradient at r to the luminosity and is a second important equation for understanding stellar equilibrium, equal in importance to (2.3).

To understand where κ comes from we need to know something about the history of a photon trying to leave the core of the star. Consider an particle (a photon) hitting a slab of material (the plasma), of area A and thickness d containing N particles, consisting of particles (electrons and protons) each presenting area σ to the incoming particle (σ is called the cross-section). The probability of the projectile hitting a slab particle is

$$
\begin{aligned}
\mathcal{P} &= \frac{N\sigma}{A} \\
&= \frac{N\sigma d}{V} \\
\Rightarrow \qquad \frac{\mathcal{P}}{d} &= \frac{N\sigma}{V} = n\sigma,
\end{aligned}
$$

where $V = Ad$ is the volume of the slab. Define the *mean free path* of one of the incoming particles as being the average distance they travel before being scattered. This can be estimated as the value of d at which $\mathcal{P} = 1$

$$
l = \frac{1}{n\sigma}.
$$

The opacity κ is inversely proportional to the mean free path and is defined as

$$
\kappa = \frac{1}{\rho l} = \frac{n\sigma}{\rho}. \tag{2.22}
$$

As was done for the dynamical equation for stellar equilibrium are (2.3), where the density can be eliminated to give a second order differential equation for the pressure, (2.20) can be used to eliminate the luminosity from (2.9.1) to give a second order equation for the temperature, leading to

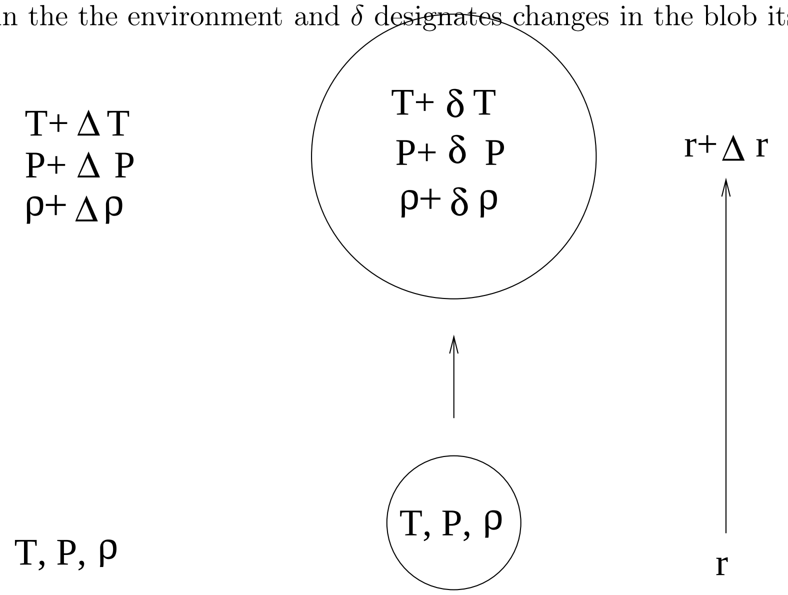
$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho(r) \tag{2.23}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\kappa\rho}\frac{dT^4}{dr}\right) = -\frac{3}{ac}w(r). \tag{2.24}$$

We need to know equations of state $P(\rho, T)$ and $w(\rho, T)$ and $\kappa(\rho, T)$ to completely specify the system. A full understanding of the solutions requires a detailed knowledge of the opacity, which depends sensitively on the chemical composition of the star. This can only be achieved numerically and will not be taken further here.

2.9.2 Convection

The above equations assume that it is radiation that dominates energy transport, but this is not always true. Consider a rising blob of gas in a star, from r to $r + \Delta r$. (Δ designates changes in the the environment and δ designates changes in the blob itself).



Assume:

1. the pressure equalizes immediately

$$\Delta P = \delta P.$$

2. ideal gas law for the star: $P = \frac{\rho}{m} k_B T$, $\Delta P = \frac{k_B}{m}(T\Delta\rho + \rho\Delta T)$

$$\Rightarrow \qquad \frac{\Delta P}{P} = \frac{\Delta\rho}{\rho} + \frac{\Delta T}{T}.$$

3. the blob expands adiabatically (*i.e.*without heat energy being exchanged between the blob and its environment): $PV^\gamma = const$ or $P\rho^{-\gamma} = const$ ($\gamma = 5/3$ for a monatomic gas).

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} = \frac{\Delta P}{P}.$$

The blob will continue to rise and convection sets in if $\delta\rho < \Delta\rho$ while for $\delta\rho > \Delta\rho$ it will stop rising and start sinking. So there will be no convection if

$$\frac{1}{\gamma}\frac{\Delta P}{P} = \frac{\delta\rho}{\rho} > \frac{\Delta\rho}{\rho} = \frac{\Delta P}{P} - \frac{\Delta T}{T}$$
$$\Rightarrow \qquad \frac{\Delta T}{T} > \left(1 - \frac{1}{\gamma}\right)\frac{\Delta P}{P} \qquad \Rightarrow \qquad \frac{dT}{dr} > \left(\frac{\gamma - 1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$
$$\Rightarrow \qquad \left|\frac{dT}{dr}\right| < \left(\frac{\gamma-1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|$$

where the limit $\Delta r \to 0$ has been taken (both dT/dR and dP/dr are negative and $\gamma > 1$).

The star is stable against convection, and energy is transmitted by radiation, if

$$\left|\frac{dT}{dr}\right|_{Radiation} < \left(\frac{\gamma-1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|.$$

So, assuming quasi-static equilibrium and using equation (2.3) with $\ddot{r} = 0$,

$$\frac{3\kappa\rho}{16\pi ac}\frac{L}{r^2T^3} < \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$
$$\Rightarrow \qquad \frac{3\kappa\rho}{16\pi ac}\frac{L}{r^2T^3} < \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{GM\rho}{r^2}$$
$$\Rightarrow \qquad \frac{3\kappa}{16\pi ac}L < \frac{\gamma-1}{\gamma}\frac{T^4}{P}GM.$$

Conversely convection is more important than radiation for energy transport, if

$$\frac{L(r)}{M(r)} > \frac{\gamma-1}{\gamma}\frac{T^4G}{P}\frac{16\pi ac}{3\kappa} = \frac{\gamma-1}{\gamma}\frac{P_{Radiation}}{P}\frac{16\pi Gc}{\kappa} := \left.\frac{L}{M}\right|_{critical} \tag{2.25}$$

where $P = P_{Radiation} + P_{Gas}$ is the total pressure. In the centre of the Sun we saw earlier that $P_{Gas} >> P_{Radiation}$, so $P = P_{Gas}$ in the solar centre. For example, using realistic values for the central core of the Sun, take a monatomic gas $\gamma = 5/3$ with $P_c = 2.5 \times 10^{16} Nm^{-2}$ and $T_c = 1.6 \times 10^7 K$, $\kappa_c = 0.1 m^2kg^{-1}$ (a value that will be derived in the next section)

$$\left.\frac{L}{M}\right|_{critical} = 3 \times 10^{-3} Js^{-1}kg^{-1}.$$

Compare this to the power per unit mass arising from Hydrogen burning in the solar core (2.16)

$$\frac{w_c}{\rho_c} = 1.4 \times 10^{-3} Js^{-1}kg^{-1}. \tag{2.26}$$

Model the Sun as a central Hydrogen burning core of radius R_c with constant w_c and ρ_c, surrounded by an envelope in which there is no nuclear burning so $w = 0$ in the envelope. Then, for the core

$$\frac{L(R_c)}{M(R_c)} = \frac{4\pi R_c^3 w_c/3}{4\pi R_c^3 \rho_c/3} = \frac{w_c}{\rho_c} < \left.\frac{L}{M}\right|_{critical}$$

and we conclude that the centre of the Sun is not convective.

The observed value at the surface:

$$\frac{L_\odot}{M_\odot} = \frac{3.8 \times 10^{26}}{2 \times 10^{30}} = 1.9 \times 10^{-4} Js^{-1}kg^{-1} \tag{2.27}$$

is an order of magnitude less than (2.26) but the radiation pressure at the surface is also decreased, by the much greater factor of $(T_\odot/T_c)^4 = (5.8 \times 10^3)^4/(1.6 \times 10^7)^4 = 1.7 \times 10^{-14}$, so that (2.27) is greater than $(L/M)_{critical}$ near the surface: the outer layers of the Sun *are* convective. In fact the Sun becomes convective about 5/7 of the way out from the centre: energy is transferred by radiation, and equation (2.9.1) is valid, from the centre out to about $0.7R_\odot$ and energy is transferred by convection the rest of the way, where equation (2.9.1) is not valid. This is due in part to the fact that gas pressure dominates radiation pressure in the core but we saw earlier that this is not true for more massive stars — stars a few times the mass of the Sun are convective in the core and equation (2.9.1) is not applicable to such stars. The theory of convective energy transport is not as simple as that of radiation transport and will not be pursued further here.

2.10 Mass-Luminosity Relation

Nuclear reactions in stars lead to typical internal temperature $T_c \approx 10^7 K$. If this were the surface temperature the luminosity of the Sun would be

$$L_{naive} = 4\pi R_\odot^2 \sigma_{SB} T_c^4 = 2.3 \times 10^{40} J/s, \tag{2.28}$$

using $T_c = 1.6 \times 10^7 K$, whereas the observed solar Luminosity is

$$L_\odot = 3.8 \times 10^{26} J/s \tag{2.29}$$

because surface temperature is actually $T_\odot = 5800K \ll T_c$. What causes this discrepancy?

A thermal photon trying to escape from the core scatters off plasma particles[3] with a mean free path $l = \frac{1}{\rho\kappa}$, the photon performs what is called a *random walk*. Consider a

[3] electrons are more important than protons in this regard because the cross-section is inversely proportional to the the square of the target's mass

photon taking k steps, each of the same length l but in random directions

$$\vec{D} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_k$$

$$D^2 = \vec{l}_1.\vec{l}_1 + \vec{l}_2.\vec{l}_2 + \ldots + \vec{l}_k.\vec{l}_k + 2\sum_{i<j}^{k} \vec{l}_i.\vec{l}_j$$

For k large $\sum_{i<j}^{k} \vec{l}_i.\vec{l}_j = 0$ because the directions are random

$$D^2 = kl^2 \qquad \Rightarrow \qquad D = \sqrt{k}l$$

For a photon to escape from the centre of the Sun to the surface: $D = R_\odot$,

$$k = \frac{R_\odot^2}{l^2}$$

With no scattering, the photon would travel from the centre to the surface in a straight line at the speed of light in a time $\tau_0 = \frac{R_\odot}{c}$. Scattering retards the photon: the distance it has to travel is actually $kl = \frac{R_\odot^2}{l}$. The photon is retarded and actually takes a time

$$\tau = \tau_0 \frac{R_\odot}{l}$$

to reach the surface. This retardation decreases the power radiated by a factor $\frac{l}{R_\odot}$, since a power is an energy divided by time. Since energy is conserved these two powers are related by a ratio of times,

$$L_\odot \approx \frac{\tau_0}{\tau} L_{naive} \approx \frac{l}{R_\odot} L_{naive}.$$

So

$$l \approx R_\odot \frac{L_\odot}{L_{naive}} \approx R_\odot \left(\frac{T_\odot}{T_c}\right)^4 \approx (7 \times 10^8) \times \left(\frac{6000}{10^7}\right)^4 \approx 10^{-4} m$$

Since the photon has to travel a distance $R_\odot^2/l >> R_\odot$ it takes a time

$$\tau \approx \frac{R_\odot^2}{cl} \approx 5 \times 10^5 years$$

to get to the surface! As long as the Sun is in equilibrium the total power radiated is the same at the core as at the surface, $L_\odot = 3.8 \times 10^{26}\ W$, the core is very much hotter because it is well insulated.

This can be used to estimate the opacity κ_c in the centre of the Sun. Generally κ_c depends rather sensitively on the temperature and the chemical composition, but a rough estimate can be obtained using $l \approx 10^{-4} m$ and $\rho_c = 1.6 \times 10^5 kg \ \Rightarrow$

$$\kappa_c = \frac{1}{l\rho_c} \approx 10^{-1} m^2 kg^{-1}. \qquad (2.30)$$

We can now derive a relation between the surface luminosity and the mass of a typical star, such as the Sun. In a simple model with uniform density $T_c = \frac{1}{4}\left(\frac{4\pi}{3}\right)^{1/3} \frac{Gm_p}{k_B} M^{2/3} \rho_\odot^{1/3}$

$$\begin{aligned}
L_\odot &= 4\pi R_\odot^2 \sigma_{SB} T_\odot^4 \approx 4\pi R_\odot^2 \sigma_{SB} T_c^4 \frac{l}{R_\odot} \\
&\approx 4\pi\sigma_{SB} \left(\frac{1}{4} \left(\frac{4\pi}{3}\right)^{1/3} \frac{Gm_p}{k_B} M^{2/3} \rho_\odot^{1/3} \right)^4 l R_\odot \\
&\approx (4\pi)^{7/3} \left(\frac{1}{4}\right)^4 \left(\frac{1}{3}\right)^{4/3} \sigma_{SB} \left(\frac{Gm_p}{k_B}\right)^4 M^{8/3} \rho_\odot^{4/3} R_\odot l
\end{aligned}$$

using $M_\odot \approx \frac{4\pi}{3}\rho_\odot R_\odot^3 \Rightarrow R_\odot \approx \left(\frac{3}{4\pi}\right)^{1/3} M^{1/3} \rho_\odot^{-1/3}$ the magic is that the density cancels when $l \approx 1/\kappa\rho_\odot$ leading to

$$L_\odot \approx \frac{\pi^2}{48} \sigma_{SB} \left(\frac{Gm_p}{k_B}\right)^4 \frac{M^3}{\kappa}$$

independent of ρ, where κ is an average opacity in the star. Ignoring approximate numerical factors we get the Mass-Luminosity relation

$$\boxed{L_\odot \approx \sigma_{SB} \left(\frac{Gm_p}{k_B}\right)^4 \frac{M^3}{\kappa}} \tag{2.31}$$

Observationally $L \propto M^3$ only works for very massive stars, main sequence stars satisfy $L \propto M^{3.5}$ while $M^{2.3}$ works better for low mass stars.

2.11 Eddington Limit

This is an upper limit on the luminosity a star can have for a given mass because radiation pressure can blow a star apart!

The radiation flux was given as

$$\mathcal{F}(r) = -\frac{c}{\rho\kappa}\frac{dP_{Rad}}{dr} = -\frac{c}{n\sigma}\frac{dP_{Rad}}{dr} = \frac{L(r)}{4\pi r^2}.$$

$\frac{dP_{Rad}}{dr}$ has dimensions of force/volume so define a force/volume, f_{Rad},

$$f_{Rad} = -\frac{dP_{Rad}}{dr} = \frac{n\sigma}{c}\mathcal{F} = \frac{n\sigma}{c}\frac{L(r)}{4\pi r^2}$$

Force/volume = acceleration × density so

$$a_{Rad} = \frac{f_{Rad}}{\rho} = \frac{\sigma}{m_p c}\frac{L(r)}{4\pi r^2} \tag{2.32}$$

is the acceleration due to Radiation pressure. The acceleration due to gravity at a distance r from the centre is

$$g = -\frac{GM(r)}{r^2}$$

so stability at the surface requires that, at $r = R$, $a_{Rad} < |g| \Rightarrow$

$$L < \frac{4\pi R^2 m_p c}{\sigma}\frac{GM}{R^2} = \frac{4\pi m_p Gc}{\sigma}M := L_{Eddington} \tag{2.33}$$

At the surface take σ to be the cross-section for a photon to scatter off an electron in free space (this is the Thomson cross-section, $\sigma_T = 6.7 \times 10^{-29} m^2$) so

$$\begin{aligned} L_{Eddington} &= \frac{4\pi m_p Gc}{\sigma_T}M \qquad (M \text{ in } kgs,\ L \text{ in } Js^{-1}) \\ \Rightarrow \qquad \frac{L_{Eddington}}{L_\odot} &= (3.3 \times 10^4)\frac{M}{M_\odot}, \end{aligned}$$

where in the last equation the luminosity and mass have been expressed in solar units, $L_\odot = 3.8 \times 10^{26}\ Js^{-1}$ and $M_\odot = 2.0 \times 10^{30}\ kg$.

2.12 Maximum Stellar Mass

Combining Mass-Luminosity relation

$$L \approx \sigma_{SB}\left(\frac{Gm_p}{k_B}\right)^4 M^3 \rho_\odot l$$

with the Eddington limit

$$L < \frac{4\pi G m_p c}{\sigma_T}M$$

gives

$$\begin{aligned} \Rightarrow \qquad \sigma_{SB}\left(\frac{Gm_p}{k_B}\right)^4 M^3 \rho_\odot l &< \frac{4\pi G m_p c}{\sigma_T}M \\ M^2 &< 4\pi \frac{k_B^4}{G^3 m_p^3}\frac{c}{\sigma_{SB}\sigma_T}\frac{1}{\rho_\odot l} \\ M^2 &< 1.8 \times 10^{65} kg \\ M &< 4.3 \times 10^{32} kg \approx 200 M_\odot. \end{aligned} \tag{2.34}$$

2.13 Minimum Stellar Mass

As a gas cloud (a "protostar") collapses, the temperature increases. Assume an ideal gas in quasi-static equilibrium. Temperature at the core of the cloud $T_c = \frac{m_p P_c}{2 k_B \rho_c}$ (no factor 1/2 for neutral hydrogen atoms). Eventually T_c gets high enough to ionize Hydrogen, and as T_c increases further either it gets high enough for nuclear reactions to switch on ($10^6 K$), or the density gets high enough for degeneracy pressure due to electrons to halt contraction.

Assume non-relativistic electrons, degeneracy pressure

$$P = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3}$$

where n_e is the number density of electrons ($n_e = n_p = \rho/m_P$ if the star is pure Hydrogen). Total core pressure

$$P_T = \frac{2\rho_c k_B T_c}{m_p} + \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \frac{\rho_c^{5/3}}{m_p^{5/3}}$$

For our simple model ($\rho = const$) we had a core pressure of $P_T = \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$. Therefore

$$\frac{2\rho_c k_B T_c}{m_p} + \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3} = \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$$

Solving for $k_B T$ and using $n_e = n_p = \rho_c/m_p$ gives

$$\begin{aligned} k_B T &= \frac{1}{2}\left(\frac{\pi}{6}\right)^{1/3} G m_p M^{2/3}\rho_c^{1/3} - \frac{1}{2}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \frac{\rho_c^{2/3}}{m_p^{2/3}} \\ &:= A\rho_c^{1/3} - B\rho_c^{2/3}. \end{aligned}$$

Maximum temperature requires $\frac{dT}{d\rho_c} = 0$

$$\begin{aligned} &\frac{1}{3}A\rho_c^{-2/3} - \frac{2}{3}B\rho_c^{-1/3} = 0 \\ \Rightarrow \qquad &\rho_c^{-1/3} = \frac{2B}{A} \qquad \Rightarrow \qquad \rho_c = \frac{1}{8}\left(\frac{A}{B}\right)^3, \end{aligned}$$

$$\begin{aligned} k_B T^{max} &= A\frac{A}{2B} - B\left(\frac{A}{2B}\right)^2 = \frac{A^2}{4B} \\ &= \frac{\left(\frac{\pi}{48}\right)^{2/3} G^2 m_p^2 M^{4/3}}{2\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} \frac{1}{m_p^{2/3}}} \\ &= \frac{5}{2^{1/3}}\left(\frac{\pi}{6}\right)^{4/3} \frac{G^2 m_p^{8/3} M^{4/3} m_e}{h^2} \\ T_c^{max} &\propto M^{4/3}. \end{aligned} \tag{2.35}$$

Fusion requires $T_c^{max} \geq 1.5 \times 10^6 K$ so

$$M > M_{min} = \left(\frac{\frac{2^{1/3}}{5}\left(\frac{6}{\pi}\right)^{4/3} h^2 k_B \times (1.5 \times 10^6)}{G^2 m_p^{8/3} m_e}\right)^{3/4} \approx 2 \times 10^{28} kg = \frac{M_\odot}{100}$$

In summary it has been shown that stellar masses should all lie in the range

$$\boxed{0.01 M_\odot < M < 200 M_\odot}$$

Observationally

$$0.08 M_\odot < M < 150 M_\odot .$$

2.14 Degenerate Stars

2.14.1 White Dwarves

When nuclear fuel runs out, a star cools by radiation and thermal pressure can no longer sustain it against gravitational collapse. It collapses until degeneracy pressure becomes dominant. For a degenerate gas of electrons the pressure is

$$P = K n_e^\gamma = \begin{cases} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e} n_e^{5/3} & \text{non relativistic electrons} \\ \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4} n_e^{4/3} & \text{relativistic electrons} \end{cases}$$

The Fermi momentum is $p_F = h\left(\frac{3n_e}{8\pi}\right)^{1/3}$. The mass of an atomic nucleus is $m_N = Am_p$, where A is the number of neutrons plus the number of protons in the nucleus. Assuming full ionisation the electron density is $n_e = Zn_N$ where the atomic number Z is the positive charge on the nucleus. Therefore $\rho = m_N n_N = (Am_p)(\frac{n_e}{Z}) = \frac{m_p n_e}{\mu}$ where $\mu = \frac{Z}{A} \approx \frac{1}{2}$ for most stable atomic nuclei. So

$$n_e = \frac{\mu\rho}{m_p} \quad \Rightarrow \quad P = K\left(\frac{\mu\rho}{m_p}\right)^\gamma \tag{2.36}$$

i) Non relativistic: our simple model with uniform density gave pressure $P_c = \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho^{4/3}$. If P_c were due to electron degeneracy pressure

$$\begin{aligned} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_e}\left(\frac{\mu\rho}{m_p}\right)^{5/3} &= \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho^{4/3} \\ \frac{1}{2^{2/3}}\left(\frac{3}{4\pi}\right)^{2/3}\rho^{1/3} &= \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3}\frac{5m_e}{h^2}\left(\frac{m_p}{\mu}\right)^{5/3} GM^{2/3} \\ \rho^{1/3} &= \frac{1}{2^{1/3}}\frac{4\pi}{3}\frac{5m_e G}{h^2}\left(\frac{m_p}{\mu}\right)^{5/3} M^{2/3} \qquad (2.37) \\ \rho &= \frac{1}{2}\left(\frac{20\pi m_e G}{3h^2}\right)^3\left(\frac{m_p}{\mu}\right)^5 M^2 \\ &= \frac{(6.2\times 10^8)}{\mu^5}\left(\frac{M}{M_\odot}\right)^2 kg\ m^{-3}. \qquad (2.38) \end{aligned}$$

With $\mu = 1/2$

$$\rho = 2\times 10^{10}\left(\frac{M}{M_\odot}\right)^2 kg\ m^{-3}. \tag{2.39}$$

Check: for consistency ρ must give non relativistic electrons, which requires $p_F << m_e c$ for realistic values of M.

$$\begin{aligned}
\frac{p_F}{m_e c} &= \frac{h\left(\frac{3n_e}{8\pi}\right)^{1/3}}{m_e c} = \frac{h}{m_e c}\left(\frac{3}{8\pi}\frac{\mu\rho}{m_p}\right)^{1/3} \\
&= \frac{h}{m_e c}\left(\frac{3}{8\pi}\frac{\mu}{m_p}\right)^{1/3}\frac{1}{2^{1/3}}\frac{4\pi}{3}\frac{5m_e G}{h^2}\left(\frac{m_p}{\mu}\right)^{5/3} M^{2/3} \qquad \text{(using (2.37))} \\
&= \frac{20G}{2^{1/3}hc}\frac{\pi}{3}\left(\frac{3}{8\pi}\right)^{1/3} M^{2/3}\left(\frac{m_p}{\mu}\right)^{4/3} \\
&= \frac{5G}{hc}\left(\frac{2\pi}{3}\right)^{2/3}\left(\frac{m_p}{\mu}\right)^{4/3} M^{2/3} \\
&= \frac{0.9}{\mu^{4/3}}\left(\frac{M}{M_\odot}\right)^{2/3}. \qquad (2.40)
\end{aligned}$$

This ratio must be significantly less than one for the non-relativistic degeneracy pressure to be valid, for example $\mu = 1/2 \;\Rightarrow\; M << 0.3M_\odot$

ii) For $M \approx 0.3M_\odot$ and larger electrons are relativistic.

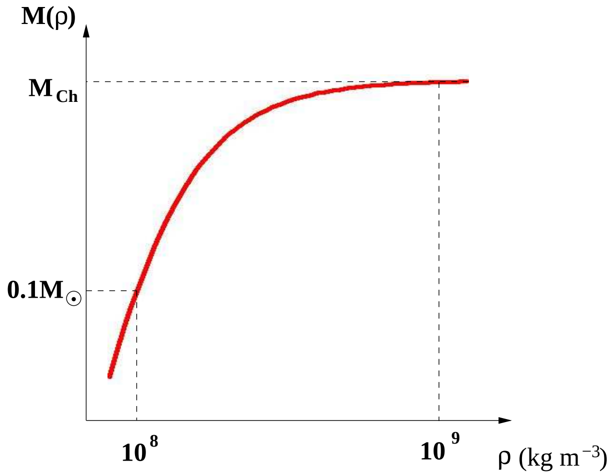
$$\begin{aligned}
\left(\frac{3}{8\pi}\right)^{1/3}\frac{hc}{4}\left(\frac{\mu\rho}{m_p}\right)^{4/3} &= \frac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3} GM^{2/3}\rho^{4/3} \\
\frac{1}{2^{2/3}}\left(\frac{3}{2\pi}\right)^{1/3}\frac{hc}{4}\left(\frac{\mu}{m_p}\right)^{4/3} &= \frac{1}{2^{2/3}}\left(\frac{2\pi}{3}\right)^{1/3} GM^{2/3} \\
M^{2/3} &= \frac{hc}{4G}\left(\frac{3}{2\pi}\right)^{2/3}\left(\frac{\mu}{m_p}\right)^{4/3}
\end{aligned}$$

$$M = \frac{3}{16\pi}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{\mu}{m_p}\right)^2 := M_{Chandraskhar} = 1.7\mu^2 M_\odot. \qquad (2.41)$$

This has assumed uniform density and better models imply

$$M_{Chandrasekhar} \approx 5.8\mu^2 M_\odot.$$

This is independent of the density. Thus for low density (non-relativistic) $M \propto \sqrt{\rho}$ while for higher densities (relativistic) M is independent of ρ. A graph of M as a function of ρ looks something like this:



There are no stable stars held up by electron degeneracy pressure for $M > M_{Ch}$. More sophisticated models (numerical integration with $\rho(r)$) $\Rightarrow$ $M_{Ch} = 5.8\mu^2 M_\odot \approx 1.4 M_\odot$ for $\mu = 1/2$. Stars with $M > M_{Ch}$ collapse catastrophically when their fuel runs out to a small dense core and rebound as a Supernova (type II), leaving behind a neutron star or a black-hole.

For $M < M_{Ch}$ the star is stable and very small — a *White Dwarf*, held up by electron degeneracy pressure. To calculate the radius consider *e.g.* $M = M_\odot$ (electrons are relativistic). Then $p_F = m_e c$, $\Rightarrow m_e c = h\left(\frac{3n_e}{8\pi}\right)^{1/3} \Rightarrow n_e = \frac{8\pi}{3}\left(\frac{m_e c}{h}\right)^3$. Charge neutrality $n_e = n_p$, $\rho = \frac{m_p n_p}{\mu} = \frac{m_p n_e}{\mu}$,

$$\rho = \frac{8\pi}{3}\frac{m_p}{\mu}\left(\frac{m_e c}{h}\right)^3 \approx 2\times 10^9 kgm^{-3}$$

Assuming constant density

$$\begin{aligned} R = \left(\frac{3M_\odot}{4\pi\rho}\right)^{1/3} &= \left(\frac{3\times 2\times 10^{30}}{4\pi\times 2\times 10^9}\right)^{1/3} m \\ &\approx 0.62\times 10^7 m \\ &\approx 6200 km \\ &\approx 10^{-2} R_\odot \end{aligned}$$

hence "dwarf".
e.g. Sirius B is a "white dwarf" with $R = 0.007R_\odot = 5000km$ Numerical calculations $\Rightarrow$ white dwarves should be stable up to $\rho \approx 10^{11} kgm^{-3}$

2.14.2 Neutron stars

For $M > M_{Ch}$, the star continues to collapse until electrons and protons are compressed together and they combine and form neutrons by inverse β-decay. In free space $n \to p + e^- + \bar{\nu}_e$ (β-decay). Inverse β-decay: $e^- + p + energy \to n + \nu_e$. We then have a star consisting solely of neutrons ($M_\odot \equiv 10^{57}$ neutrons) made of "nuclear matter". For

$\rho > 1.4 \times 10^{14}\ kg/m^3$ electrons begin to fuse with protons resulting in atomic nuclei which are "neutron rich", *i.e.* have a larger ratio of neutrons to protons than usual. The excess neutrons leak out in a process known as "neutron drip" until eventually there are no protons left and no nuclei, we have a state of pure neutron matter. The entire star is a giant ball of neutrons with a density of $\rho_{nuclear} \approx 3 \times 10^{17} kgm^{-3}$.

Neutrons are fermions, they have spin 1/2, and there is therefore a degeneracy pressure associated with then. If the mass is not too high, the "star" can be supported by neutron degeneracy pressure.

1. Non relativistic:

$$P = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{5m_n} n^{5/3},$$

   e.g. $\rho = 4 \times 10^{14} kgm^{-3} \Rightarrow n = \frac{\rho}{m_n} = \frac{4\times10^{14}}{1.7\times10^{-27}} m^{-3} = 2.4 \times 10^{41} m^{-3}$. The Fermi momentum is then

$$p_F = h\left(\frac{3n}{8\pi}\right)^{1/3} = 2.0 \times 10^{-20}\ kgms^{-1} \qquad \Rightarrow \qquad \frac{p_F}{m_n c} = 0.04$$

   and neutrons are non-relativistic at these densities. The calculation exactly parallels that of white dwarves, except that $\mu = 1$ and $m_e \to m_n$ in (2.37), increasing the density by $(m_n/m_e)^3 = 6.2 \times 10^9$, giving

$$\rho = 3.8 \times 10^{18} \left(\frac{M}{M_\odot}\right)^2 kgm^{-3}$$

   for non-relativistic neutrons. Neutrons become relativistic for $M \approx M_\odot$ (equation (2.40) with $\mu = 1$), when $p_F \approx m_n c\ \Rightarrow \rho \approx 6.1 \times 10^{18}\ kg\ m^{-3}$.

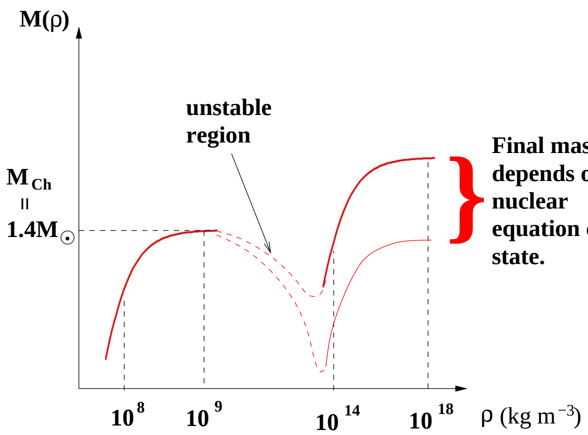
2. for $M > M_\odot$ neutrons are relativistic and we get (2.41) with $\mu = 1$

$$M = 1.7 M_\odot.$$

This ignores nuclear interactions between the neutrons and in reality the upper limit for the mass of a stable neutron star depends on the strength of the nuclear forces. It lies somewhere between $1.5M_\odot \sim 2.7M_\odot$.
The radius for $M = 1.7M_\odot$ and $\rho = 6.2 \times 10^{18} kgm^{-3}$ is:

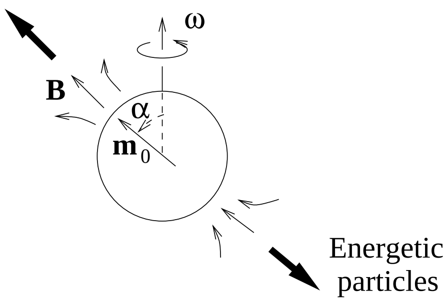
$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3} \approx 5km$$

Generically, more sophisticated calculations give $R \approx 10km$.



2.14.3 Pulsars

A pulsar is a rotating neutron star with a strong magnetic field. All stars are believed to have magnetic fields, the Sun for example has a field $B_\odot \approx 2 \times 10^{-4} T$ (compared to that of the Earth, $B_\oplus \approx 3.1 \times 10^{-5} T$). A pulsar is a rotating dipole and rotating dipoles emit electromagnetic radiation, thus losing energy. [4]



e.g. pulsar in the Crab Nebula: $\omega = 190\ radians\ s^{-1}, T = \frac{2\pi}{\omega} = 3 \times 10^{-2} s$, and it is slowing down slowly over time, $\dot{\omega} = -2.4 \times 10^{-9} s^{-2}$.

How large can ω be? ω_{max} from

$$\omega_{max}^2 R = \frac{GM}{R^2}$$
$$\omega_{max}^2 = \frac{GM}{R^3} \propto \rho$$
$$T_{min} = \frac{2\pi}{\omega_{max}} = 2\pi\sqrt{\frac{R^3}{GM}} \tag{2.42}$$

[4]This is not actually the source of the radiation that is directly observed. The electromagnetic pulses seen from a pulsar are due to radiation from beams of charged particles emitted along the axis of the dipole and sweeping past the Earth like a lighthouse beam.

For a neutron star with $M = M_\odot$, $R = 10km$: $T_{\min} = 2\pi\sqrt{\frac{10^{12}}{6.7\times10^{-11}\times2\times10^{30}}}s = 5\times10^{-4}s$

The total time-averaged power radiating by a rotating dipole with magnetic dipole moment m_0 is:

$$P = \frac{2}{3}\frac{\mu_0}{4\pi c^3}\omega^4 m_0^2 \sin^2\alpha.$$

The rotational energy is $E = \frac{1}{2}I\omega^2$ so we expect

$$P = -\frac{dE}{dt} = -I\omega\dot{\omega}.$$

For a sphere of constant density $I = \frac{2}{5}MR^2$

$$\dot{\omega} = -\frac{P}{I\omega} = -\frac{5}{3}\frac{\mu_0}{4\pi c^3}\frac{\omega^3}{MR^2}(m_0\sin\alpha)^2$$

For $M = M_\odot,\ R = 10km,\ \omega = 190s^{-1},\ \dot{\omega} = -2.4\times10^{-9}s^{-2}$

$$\Rightarrow \qquad (m_0\sin\alpha) = 2.7\times10^{27}Cm^2s^{-1}.$$

So the magnitude of $|\vec{B}|$ at the surface is $B \approx \frac{\mu_0}{4\pi}\frac{m_0}{R^3} = 10^{-7}\frac{(2.7\times10^{27})}{10^{12}}T = 2.7\times10^8T \approx 10^{12}B_\odot$. This is not unreasonable, if the Sun had the same dipole moment at the centre the field at the surface of the Sun would be related to that of the neutron star by a factor $\left(\frac{R}{R_\odot}\right)^3 \approx 1/(7\times10^4)^3 \approx 3\times10^{-15}$, which is not too far off.

We now have a relation between ω and $\dot{\omega}$, $\dot{\omega} = -A\omega^3$ with $A = \frac{5}{2}\frac{\mu_0}{4\pi c^3}\frac{(m_0\sin\alpha)^2}{MR^2}$

$$\frac{d\omega}{dt} = -A\omega^3$$
$$\frac{d\omega}{\omega^3} = -Adt$$
$$-\frac{1}{2}(\omega^{-2} - \omega_0^{-2}) = -A(t - t_0)$$

where ω_0 = initial period at time t_0. Choose the time of formation $t_0 = 0$, then the current age is

$$t = \frac{1}{2A}\left(\frac{1}{\omega^2} - \frac{1}{\omega_0^2}\right) < \frac{1}{2A}\frac{1}{\omega^2} = \frac{1}{2}\frac{\omega}{|\dot{\omega}|}$$

giving an upper bound on the age.
For the crab pulsar $t < \frac{1}{2}\frac{\omega}{|\dot{\omega}|} = \frac{1}{2}\frac{190}{2.4\times10^{-9}}s = 4\times10^{10}s = 1250yr$. The pulsar is known to be 956 years old, the supernova was observed in 1054 AD.

2.14.4 Black Holes

If the final mass of a dead star $M > 3M_\odot$ it collapses even further and R decreases. Naïvely the escape velocity from the surface is given by $\frac{1}{2}v^2 = \frac{GM}{R}$. For $v = c$

$$R = \frac{2GM}{c^2} := R_S \tag{2.43}$$

called the Schwarzschild-radius. This argument is incorrect for relativistic velocities, but the conclusion is correct. If $R < R_S$ light cannot escape and the star forms a black hole. e.g. for $M = M_\odot$: $R_S \approx 3km$.
Young black holes are often surrounded by matter which spirals in and heats up $\Rightarrow$ bright X-ray sources or intense radio sources. e.g. Cygnus X-1 is an intense X-ray source showing fluctuations on a time-scale $10^{-3}s \Rightarrow R < 300km$ and yet $M >> M_\odot$; Sagittarus A* is a compact mass at the centre of the Galaxy with $M \approx 2.6 \times 10^6 M_\odot$ which is believed to be a black-hole.
For a given mass the density is $\rho = \frac{3M}{4\pi R^3} = \frac{3c^2}{8\pi G R^2} \propto \frac{1}{R^2} \propto \frac{1}{M^2}$ e.g. for one solar mass $M = M_\odot$ a neutron star would have $R \approx 10km, \rho \approx 10^{18} kgm^{-3}$, while a black hole would have $R \approx 3km, \rho \approx 3 \times 10^{19} kgm^{-3}$. For $M = 10^8 M_\odot$ (e.g. Andromeda Galaxy: central black hole with $M = 3 \times 10^7 M_\odot$) $\rho \approx \frac{10^{19}}{10^8} kgm^{-3} = 10^3 kgm^{-3} = 1\ g/cc$, about the density of water!

2.15 Active Galactic Nuclei (AGN) and Quasars (QSO)

These are extremely energetic:

$$L \approx (10^{10} - 10^{13}) L_\odot \approx (10^{37} - 10^{40}) Js^{-1}$$

Extra galactic (extremely distant):

$$d \approx 10^9 lyr$$

They are believed to be very young galaxies. The AGN record is $L = 10^{41} Js^{-1}$ but more typically

$$L = 10^{12} L_\odot \quad \text{Eddington limit:} \quad \Rightarrow \quad M \geq 3 \times 10^8 M_\odot$$

The power radiated is equivalent to loss of mass:

$$|\dot{M}| = \frac{10^{40} Js^{-1}}{c^2} = \frac{10^{40}}{(3 \times 10^8)^2} kgs^{-1} \approx 10^{23} kgs^{-1} \approx \frac{1}{2 \times 10^7} M_\odot s^{-1} \approx M_\odot yr^{-1}.$$

Nuclear burning is only $\approx 1\%$ efficient and it does not seem possible to produce these kinds of energies by nuclear forces. For the Sun the power per unit mass was $\frac{w}{\rho} = 1.4 \times 10^{-3} W kg^{-1}$. An AGN or QSO with $M \approx 10^9 M_\odot$ could only produce

$$10^9 \times (2 \times 10^{30}) \times (1.4 \times 10^{-3}) Js^{-1} = 2.8 \times 10^{36} Js^{-1}$$

The energy source cannot be nuclear energy. The only other known possible source of energy is gravity. The gravitational energy of a mass M in radius R is $|E_{Grav}| = \frac{GM^2}{R}$. For a black hole

$$R = \frac{2GM}{c^2} \Rightarrow |E_{Grav}| = \frac{1}{2} M c^2$$

Using $M \approx 10^9 M_\odot$ and $|\dot{E}| \approx 10^{40}\ Js^{-1}$ gives $\frac{|\dot{E}|}{|E_{Grav}|} \approx 10^9\ yrs$, or about 1/10th the age of the Universe. Assuming that the gravitational force is the source of power for AGN's explains why there is none observed very close the our own Galaxy — there is none left today, we can only see them from a time much earlier than the present day.

3 Cosmology and Expansion of the Universe

On very large length scales ($> 100Mpc$) the distribution of galaxies appears to be isotropic. If we assume that all points in space are equivalent, *i.e.* we are not at a special point (this is called the *Copernican Principle*), then the distribution must be is isotropic about all other points and this actually implies that distribution of galaxies is homogeneous on very large scales, *i.e.* it is uniform. Therefore mass density ρ is independent of position on large enough length scales.

3.1 Cosmological Constant

Consider a spherical shell of matter, radius R and thickness δR, of mass $\delta m = 4\pi R^2 \delta R \rho$ around a mass $M(R) = \frac{4\pi R^3 \rho}{3}$. In Newtonian Gravity, the dynamics of the shell is the same as it would be if the central mass were concentrated at a point in the centre and any mass outside the shell doesn't affect it at all if it is spherically distributed (consequence of the inverse square force). The Newtonian equation of motion for the shell is

$$\delta m \ddot{R} = -\frac{G\delta m M}{R^2}. \tag{3.1}$$

We shall assume the following:

1. $\dot{R} \ll c$ (non-relativistic velocities);
2. $\frac{2GM}{c^2 R} = \frac{8\pi G}{3c^2} R^2 \rho \ll 1$ (weak gravitational field, Newtonian Gravity is valid);
3. $R \gg$ galactic separation, so we approximate ρ by a smooth function $\rho(R)$.

It might be thought that 1) and 2) are related by the virial theorem $\Rightarrow \frac{\dot{R}^2}{c^2} \approx \frac{2GM}{c^2 R}$, but the virial theorem only applies to stable gravitationally bound systems and is not applicable to galaxies that are so far apart that they are not bound to each other by gravitational forces.

2) and 3) $\Rightarrow$ *galactic separation* $\ll R \ll \sqrt{\frac{3}{8\pi G \rho}} c$. Provided R lies in this range we shall assume we can use equation (3.1) and immediately get the first integral

$$\frac{1}{2}\delta m \dot{R}^2 - \frac{G\delta m M}{R} = E$$
$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \frac{E}{\delta m} = \frac{E}{4\pi R^2 \delta R \rho} := \epsilon.$$

where E is the total energy of the shell and ϵ the energy of the shell per unit mass, which is a constant if δm is constant and has dimensions of *velocity*^{2}. Assuming uniform density M can be eliminated in favour of ρ,

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}GR^2\rho = \epsilon. \tag{3.2}$$

This equation was derived using Newtonian gravity and non-relativistic physics. In relativistic physics Newton's $1/R^2$ force is invalid (it's only valid for small velocities). We should use the full power of Einstein's General theory to determine the dynamics of the Universe and this lies outside the scope of this course. Remarkably General Relativity gives the same equation but with a different interpretation. In General Relativity R is not a radial co-ordinate, it is a length scale determining the physical size of lengths in 3-dimensional space. If $R = R(t)$ then when $\dot{R} > 0$ space is said to be expanding while when $\dot{R} < 0$ space is contracting. We can interpret R as the physical distance between any two fixed galaxies, provided their separation is of the order of $100MPc$, or more.

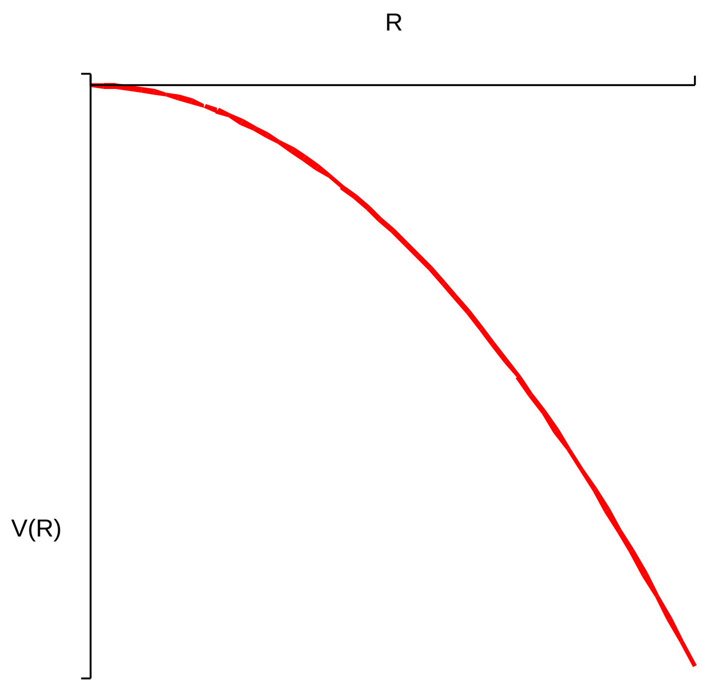
Equation (3.2) is the same as the energy of a point particle of unit mass moving on the half-line $R > 0$ in a quadratic potential, like a harmonic oscillator equation but with a *negative* co-efficient. Suppose $\rho = \rho_0$ is independent of R. Define the constant

$$\Lambda := \frac{8\pi G\rho_0}{c^2},$$

which has dimensions of $(length)^2$, then

$$\frac{1}{2}\dot{R}^2 - \frac{\Lambda}{6}R^2 = \epsilon \tag{3.3}$$

(the 1/6 is a standard convention in cosmology). We can get a qualitative understanding of the behaviour simply by plotting the potential $V(R) = -\frac{\Lambda}{6}R^2$,



If the energy $\epsilon > 0$ then R can have any value in the range $0 < R < \infty$, with $|\dot{R}|$ increasing with R; if $\epsilon < 0$ then there is a repulsive barrier and R cannot reach zero. Neither of these cases allows for a static solution with $\dot{R} = 0$. A static solution is only possible when $\epsilon = 0$, which allows for $R = \dot{R} = 0$, but this is clearly unstable — the slightest deviation away from $R = 0$ and R will start to grow, eventually reaching infinity. We conclude from this that $\rho = const$ does not allow for and stable static solution: on sufficiently large scales the distances between galaxies cannot be constant, so ρ must change with time, the Universe is either expanding or contracting — observationally it is expanding and $R(t)$ is growing as a function of time.

Consider two galaxies a distance R apart with M being the total amount of mass in a sphere of radius R centred on one of the galaxies. Even if R is changing with time it is reasonable to assume, if mass cannot be created or destroyed, that M is constant. In that case

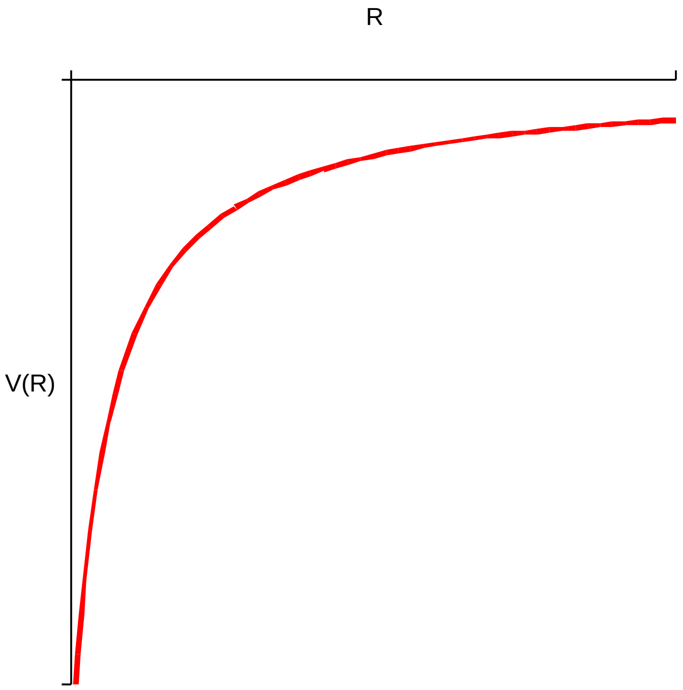
$$M = \frac{4\pi\rho}{3}R^3 \qquad \Rightarrow \qquad \rho(R) = \frac{3M}{4\pi}\frac{1}{R^3}.$$

Note that, with the interpretation of R as being the physical distance between the galaxies, $\rho \propto 1/R^3$ does not mean that ρ depends on position — ρ is independent of position but is a decreasing function of time if $R(t)$ is an increasing function of time.

For notational convenience let $A = 2GM$, then equation (3.2) gives

$$\frac{1}{2}\dot{R}^2 - \frac{A}{2R} = \epsilon. \tag{3.4}$$

This equation has the same mathematical form as that of the energy of a projectile with unit mass thrown vertically upward from the surface of the Earth, moving in a potential $V(R) = -\frac{A}{2R}$,

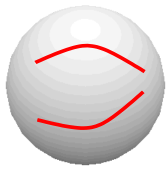


There is an attractive force towards $R = 0$: if $\dot{R} > 0$ initially the late time behaviour depends on the sign of ϵ. ϵ has dimensions of $(velocity)^2$ so define a dimensionless parameter K by $\epsilon = -\frac{1}{2}Kc^2$. Then there are three types of behaviour:

$K < 0$	R increases indefinitely and $\dot{R} > 0$ always;
$K > 0$	R reaches a maximum value and decreases again;
$K = 0$	R increases indefinitely but $\dot{R} \to 0$ as $t \to \infty$.

$K = 0$ is a critical value, corresponding to the notion of escape velocity in Newtonian dynamics but the physical interpretation of K in General Relativity is very different. K is a measure of the *curvature* of 3-dimensional space.

- $K = 0$ is flat Euclidean space;
- $K > 0$ is a 3-dimensional analogue of the surface of a 2-dimensional sphere. In a space with positive K, the trajectories of two projectiles will bend toward each other, rather like the paths of two ships following great circles on the surface of the Earth.



Mathematically a 3-dimensional sphere of radius r can be described by imposing the constraint $v^2 + x^2 + y^2 + z^2 = r^2$ on Cartesian co-ordinates (v, x, y, z) in flat 4-dimensional Euclidean space. This is a natural extension of the geometry of a circle in 2-dimensional Euclidean space with Cartesian co-ordinates (y, z), $y^2 + z^2 = r^2$ (a "1-dimensional sphere"), and the usual 2-dimensional sphere, $x^2 + y^2 + z^2 = r^2$;

- $K < 0$ is a 3-dimensional space in which trajectories *diverge*. It is a 3-dimensional analogue of a hyperbola $y^2 - z^2 = r^2$ in 2-dimensional Euclidean space, and can be described by imposing the constraint $v^2+x^2+y^2-z^2 = r^2$ on Cartesian co-ordinates (v, x, y, z) in flat 4-dimensional Euclidean space.

The full story appears to be that, at the present time, the mass density of the Universe at length scales of $100MPc$ and greater appears to have two components, one with $\rho = const$ corresponding to a constant mass density, and one with $\rho \propto 1/R^3$ corresponding to a constant amount of mass in a sphere of radius $R(t)$. Denoting the latter by ρ_{Mat} a combination of these two possibilities can produce a static solution. Let

$$\rho = \rho_{Mat} + \rho_\Lambda = \tfrac{3A}{8\pi G R^3} + \Lambda c^2/8\pi G$$

where ρ_{Mat} is the density of ordinary matter. Then

$$\begin{aligned} \dot{R}^2 &= \frac{8\pi G}{3}\rho_{Mat}R^2 - Kc^2 + \frac{\Lambda c^2 R^2}{3} \\ \Rightarrow \qquad \dot{R}^2 &= \frac{A}{R} - Kc^2 + \frac{\Lambda c^2 R^2}{3} \\ \Rightarrow \qquad 2\ddot{R} &= -\frac{A}{R^2} + \frac{2\Lambda c^2 R}{3} \end{aligned}$$

Choosing $\Lambda = \frac{3A}{2c^2R^3}$ (a repulsive force) gives a solution with no accelaration and

$$\dot{R}^2 = \frac{A}{R} - Kc^2 + \frac{A}{2R},$$

so $K = \frac{3A}{2c^2R} > 0$ gives a static solution with R a constant. (called Einstein's static universe, in which space is a 3-dimensional sphere with finite volume). This is static, but unfortunately unstable. The *cosmological constant* Λ was introduced by Einstein in order to obtain static solutions because he did not know at the time that R was changing and he assumed that it should be constant. At the present day $R(t)$ appears to be increasing but only changes very slowly, on cosmological time-scales of the order of billions of years. Nevertheless there is by now considerable observational evidence that Λ is positive.

We have arrived at the *Friedmann-Equation*

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2K}{R^2} = \frac{8\pi G\rho_{Mat}}{3} + \frac{\Lambda c^2}{3}} \tag{3.5}$$

This is a dynamical equation that determines the behaviour of the cosmological length scale $R(t)$ for a given mass density ρ_{Mat} and constants K and Λ. In the General Theory of Relativity it is valid for relativistic velocities $\dot{R} \approx c$ and for strong gravitational fields $\frac{8\pi G}{3c^2}R^2\rho_{Mat} \approx 1$, but still requires that $R >>$ galactic separations.

3.2 Redshift-distance relation

Consider two galaxies A and B with coordinates (r_A, θ_A, ϕ_A) and (r_B, θ_B, ϕ_B). Choose co-ordinates so that $\theta_A = \theta_B = \frac{\pi}{2}$ and $\phi_A = \phi_B = 0$. Take $r_A > r_B$, fixed for each galaxy (the coordinates are fixed to the galaxies and are called co-moving co-ordinates, analogous to Lagrangian coordinates). The distance between the galaxies is $R(t)(r_A - r_B)$ where $R(t)$ is the cosmological scale factor.

Consider a beam of light passing between A and B. It travels a distance $R(t)\delta r$ in the time $\delta t = \frac{R(t)\delta r}{c}$. Suppose light leaves A at time t_1 and arrives at B at time t_2

$$\int_{t_1}^{t_2} \frac{dt}{R(t)} = \frac{1}{c}\int_{r_A}^{r_B} dr. \tag{3.6}$$

At a later time light leaves A at $t_1 + \Delta t_1$, light would reach B at $t_2 + \Delta t_2$

$$\int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{R(t)} = \frac{1}{c}\int_{r_A}^{r_B} dr = \int_{t_1}^{t_2} \frac{dt}{R(t)}$$

$$\Rightarrow \int_{t_2}^{t_2+\Delta t_2} \frac{dt}{R(t)} = \int_{t_1}^{t_1+\Delta t_1} \frac{dt}{R(t)}.$$

Take Δt_1 and Δt_2 to be the inverse of optical frequency and assume $R(t) = R(t_1)$ is essentially constant between t_1 and $t_1 + \Delta t_1$ and $R(t) = R(t_2)$ between t_2 and $t_2 + \Delta t_2$ (this always is an extremely good approximation: for optical frequencies $\Delta t \approx 10^{-15}s$ while the scale factor R only changes appreciably on time scales of order $10^9 yrs \approx 3 \times 10^{16}s$), then

$$\begin{aligned} \frac{1}{R(t_1)}\Delta t_1 &= \frac{1}{R(t_2)}\Delta t_2 \\ \frac{R(t_1)}{R(t_2)} &= \frac{\Delta t_1}{\Delta t_2} = \frac{\nu_2}{\nu_1} \end{aligned} \tag{3.7}$$

If $R(t_2) > R(t_1)$ then $\nu_2 < \nu_1$, *i.e.* the light is redshifted.

Assuming $R(t)$ is a slowly varying analytic function of time it can be Taylor expanded around t_2 as

$$R(t_1) = R(t_2)\{1 - (t_2 - t_1)H + \ldots\} \qquad H = \left.\frac{\dot{R}}{R}\right|_{t_2},$$

with $(t_2 - t_1)H << 1$. Let $s_{AB} = c(t_2 - t_1)$ (time-of-flight distance) then

$$\frac{R(t_2)}{R(t_1)} \approx 1 + (t_2 - t_1)H = 1 + \frac{s_{AB}H}{c}.$$

If $t_2 = t_0$ is the present day then $H_0 := \left.\frac{\dot{R}}{R}\right|_{t_0}$ is called the Hubble constant. Suppose light left a distance galaxy at a time $t = t_1$ and arrives at our telescope at the present day $t_0 = t_2$. Then, assuming $t_0 - t << H_0^{-1}$, we have

$$z := \frac{\delta\nu}{\nu_2} = \frac{\nu_1 - \nu_2}{\nu_2} = \frac{\nu_1}{\nu_2} - 1 = \frac{R(t_0)}{R(t)} - 1 = \frac{s_{AB}H_0}{c}.$$

z is called the **redshift** of the galaxy, it is a measure of the amount by which light from a distant galaxy is shifted toward the red end of the spectrum, *i.e.* towards longer wave-lengths. So

$$z = \frac{H_0}{c} s_{AB} \propto s_{AB} \tag{3.8}$$

giving the **redshift distance relation**

$$\boxed{s_{AB} = \frac{c}{H_0} z.} \tag{3.9}$$

H_0 has dimensions of $(\text{time})^{-1}$ but is usually quoted as $kms^{-1}Mpc^{-1}$.

$$\begin{aligned} H_0 &= 73 \pm 4 kms^{-1}Mpc^{-1} \\ &= h \times 100 kms^{-1}Mpc^{-1} \quad h = 0.73 \pm 0.04. \end{aligned}$$

This means that a galaxy at a distance of 1 Mpc exhibits a redshift corresponding to a velocity of about 73 kms^{-1}. The linear relation (3.9) is only valid for "small" $(t_2 - t_1)$, i.e. $\frac{\delta\nu}{\nu} \ll 1$.

For example there is a cluster of galaxies in the constellation of Virgo (≈ 2500 galaxies) with an average redshift $c\frac{\delta\nu}{\nu} = 1150 kms^{-1} \;\Rightarrow\; \frac{\delta\nu}{\nu} = 0.00383$ giving $s_{AB} = \frac{c}{H_0}\frac{\delta\nu}{\nu} = 16Mpc = 50Mlyrs$. The current redshift record is $\frac{\delta\nu}{\nu} \approx 10$ which gives a naïve redshift distance relation of $s_{AB} \approx 4.5 \times 10^4 Mpc$, but this is not the true physical distance because $\frac{\delta\nu}{\nu} > 1$ is not small.

$\frac{1}{H_0}$ has dimensions of time: $\frac{1}{H_0} = \frac{1}{h}(3 \times 10^{17}s) = \frac{1}{h}10^{10}yr \approx 14$ billion years which is the approximates age of the universe. If $R(t)$ were linear then R would have been zero 14 billion years ago, but this is only a rough approximation. In this approximation $R_0 = \frac{1}{H_0}c = \frac{1}{h} \times 10^{26}m$ is the approximate size of the observable universe.

3.3 Friedmann Equation

Hubble discovered the linear redshift-distance relation in 1929. In recent years the observational data have become good enough to go beyond a linear approximation for $R(t)$.

The Friedmann equation (3.5) can be written as

$$H^2 + \frac{c^2 K}{R^2} = \frac{8\pi G}{3}\rho_{Mat} + \frac{\Lambda c^2}{3}$$

where ρ_{Mat} is the mass density of matter, as in stars, gas and dust in galaxies.

When $\Lambda = K = 0$,

$$\rho_{Mat}^{\Lambda=K=0} = \frac{3}{8\pi G}H^2.$$

Using present day values

$$\frac{3}{8\pi G}H_0^2 = h^2 \times (2 \times 10^{-26} kgm^{-3}).$$

Observationally, luminous galaxies contribute $\rho_{Mat} \approx 10^{-28} kgm^{-3}$.
Using observations of orbital dynamics of galaxies in clusters (Kepler's Law) $\rho_{Mat} \approx 10^{-27} kgm^{-3}$ (equivalent to about 1 proton per cubic metre). So $\rho_{Mat} < \rho_{Mat}^{\Lambda=K=0}$. Returning to the more general case with $\Lambda \neq 0$

$$\begin{aligned} H^2 &= \frac{8\pi G \rho_{Mat}}{3} - \frac{c^2 K}{R^2} + \frac{\Lambda c^2}{3} \\ \Rightarrow \qquad 1 &= \frac{8\pi G \rho_{Mat}}{3H^2} - \frac{c^2 K}{H^2 R^2} + \frac{\Lambda c^2}{3H^2} \end{aligned}$$

Define the following 3 constants using present day values H_0 and $R_0 = R(t_0)$

$$\Omega_M := \frac{8\pi G \rho_{Mat}}{3H_0^2} = \frac{A}{H_0^2 R_0^3}, \qquad \Omega_K := -\frac{c^2 K}{H_0^2 R_0^2} \qquad \text{and} \qquad \Omega_\Lambda := \frac{\Lambda c^2}{3H_0^2}.$$

Then only two of these are independent, since

$$\Omega_M + \Omega_K + \Omega_\Lambda = 1.$$

These can be related to a Taylor expansion of $R(t)$,

$$\begin{aligned} R(t) &= R_0 + (t - t_0)\dot{R}_0 + \frac{1}{2}(t - t_0)^2 \ddot{R}_0 + \dots \\ &= R_0 \left(1 + H_0(t - t_0) + \frac{1}{2}(t - t_0)^2 \frac{\ddot{R}_0}{R_0} + \dots \right), \end{aligned}$$

where, in what I hope is an obvious notation, $\dot{R}_0 = \dot{R}(t_0)$ and $\ddot{R}_0 = \ddot{R}(t_0)$. Define the *deceleration parameter*, $q_0 = -\frac{1}{2}\frac{\ddot{R}_0}{H_0^2 R_0}$, then

$$\begin{aligned}
\dot{R}^2 &= \frac{8\pi G\rho_{Mat}R^2}{3} - c^2K + \frac{\Lambda c^2 R^2}{3}, \qquad \rho_{Mat} = \frac{3}{8\pi G}\frac{A}{R^3} \\
\Rightarrow \qquad &= \frac{A}{R} - c^2K + \frac{\Lambda c^2 R^2}{3} \\
\Rightarrow \qquad 2\dot{R}\ddot{R} &= -\frac{A\dot{R}}{R^2} + \frac{2\Lambda c^2 R\dot{R}}{3} \\
\Rightarrow \qquad \frac{\ddot{R}}{R} &= -\frac{A}{2R^3} + \frac{\Lambda c^2}{3} = -\frac{4\pi G\rho_{Mat}}{3} + \frac{\Lambda c^2}{3} \\
\Rightarrow \qquad \frac{\ddot{R}}{H^2R} &= -\frac{A}{2H^2R^3} + \frac{\Lambda c^2}{3H^2} \\
\Rightarrow \qquad 2q_0 &= \frac{1}{2}\Omega_M - \Omega_\Lambda.
\end{aligned}$$

So Ω_Λ and Ω_M are directly related to the parameters in a Taylor expansion of $R(t)$.

What we actually measure is the redshift z, but we can convert from t to z. From (3.7), with $t_2 = t_0$ and $t_1 = t$, and the definition $z = \frac{\nu_1}{\nu_2} - 1$ we have $\frac{R_0}{R(t)} = 1 + z$

$$\Rightarrow \qquad dz = -\frac{R_0}{R^2}dR \qquad \Rightarrow \qquad \frac{dz}{1+z} = -\frac{dR}{R} \qquad \Rightarrow \qquad \frac{\dot{R}}{R} = -\frac{\dot{z}}{1+z}.$$

Now use this to write the Friedmann equation as

$$\begin{aligned}
\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3} - \frac{c^2K}{R^2} + \frac{\Lambda c^2}{3} &= \Omega_M H_0^2\left(\frac{R_0}{R}\right)^3 + \Omega_K H_0^2\left(\frac{R_0}{R}\right)^2 + \Omega_\Lambda H_0^2 \\
\dot{z}^2 &= (1+z)^2\left\{(1+z)^3H_0^2\Omega_M + (1+z)^2H_0^2\Omega_K + H_0^2\Omega_\Lambda\right\}.
\end{aligned}$$

This is a non-linear differential equation (no approximations) for $z(t)$ in terms of the constants H_0, Ω_M, Ω_K and Ω_Λ. Now we invoke (3.6) with B being our Galaxy, so we set $R_B = 0$, to write the distance d_A to a galaxy A at redshift z as

$$\begin{aligned}
d_A = R_0 r_A = cR_0\int_t^{t_0}\frac{dt}{R} &= c\int_t^{t_0}(1+z)dt = -c\int_t^{t_0}\frac{(1+z)dz}{\dot{z}} \\
\Rightarrow \qquad d_A &= c\int_0^z \frac{dz}{\sqrt{(1+z)^3H_0^2\Omega_M + (1+z)^2H_0^2\Omega_K + H_0^2\Omega_\Lambda}}. \qquad (3.10)
\end{aligned}$$

This is an exact non-linear redshift-distance relation.

For example

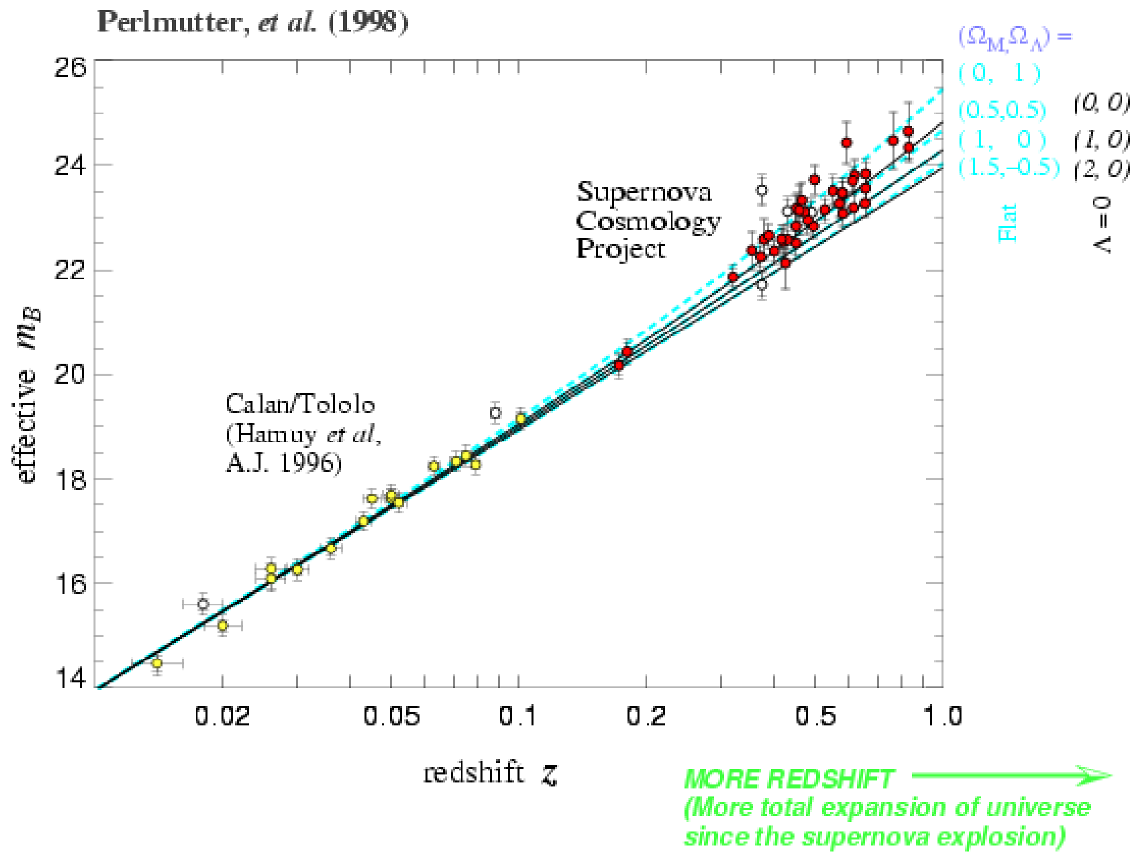
$$d_A(z) = \frac{cz}{H_0}$$

if $\Omega_\Lambda = 1$, $\Omega_M = \Omega_K = 0$, while

$$d_A(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

if $\Omega_M = 1$, $\Omega_\Lambda = \Omega_K = 0$

Although the case $\Omega_\Lambda = 1$, $\Omega_M = \Omega_K = 0$ looks very like (3.9) they are not exactly the same since the distance $d_A = R_0 r_A$ is not the same as the time-of-flight distance $s_{AB} = c(t_0 - t)$ (with $t_A = t$ and $t_B = t_0$), though their difference is negligible at small $z << 1$.



Observational data plotting galactic distance against redshift (the vertical axis is the magnitude, which is essentially the logarithm of the distance). Taken from *http://supernova.lbl.gov/*

Observationally

$$\Omega_M = 0.28 \pm 0.04 \quad \begin{cases} 0.042 \pm 0.004 & \text{"ordinary matter" (neutrons, protons)} \\ 0.24 & \text{"Dark matter", no one knows what this is} \end{cases}$$

$$\Omega_\Lambda = 0.72 \pm 0.04 \quad \text{often called "Dark Energy".}$$

$$1 = \Omega_M + \Omega_K + \Omega_\Lambda \Rightarrow \Omega_K = 0.0 \pm 0.06$$

$\Omega_K = 0 \Rightarrow 1 = \Omega_M + \Omega_\Lambda \Rightarrow q = \frac{1}{2}\left[\frac{1}{2}(1 - \Omega_\Lambda) - \Omega_\Lambda\right] = \frac{1}{4} - \frac{3}{4}\Omega_\Lambda = -0.29 \pm 0.03 < 0$ implying that the expansion rate of the Universe is accelerating.

3.4 The Friedmann Models

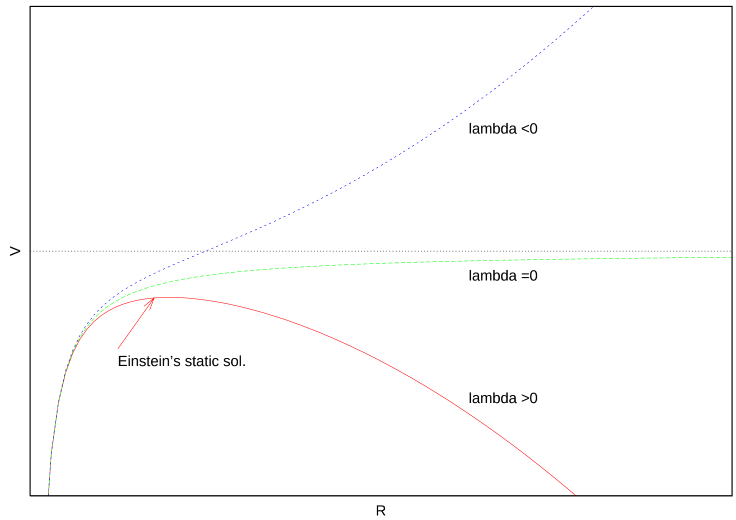
The Friedmann equation is

$$\dot{R}^2 = \frac{A}{R} - c^2 K + \frac{\Lambda c^2 R^2}{3}.$$

Pursuing the analogy with 1-dimensional particle mechanics:

$$\underbrace{\frac{1}{2}\dot{R}^2}_{\text{kinetic energy}} \underbrace{-\frac{1}{2}\left[\frac{\Lambda c^2 R^2}{3} + \frac{A}{R}\right]}_{\text{potential energy}} = \underbrace{-\frac{c^2 K}{2}}_{\text{total energy}}$$

Think of $V(R) = -\frac{A}{2R} - \frac{\Lambda c^2 R^2}{6}$ as the potential energy per unit mass of a particle moving in one dimension.



The "potential" $V(R)$ plotted as a function of R.

The behaviour of solutions depends, among other things, on the sign of K. Rescale R to set $K = \pm 1$ (or zero). Look for solutions with $\dot{R} > 0$ and $R(0)$ a non-negative constant (possibly zero). The most general case, with the three constants K, A and Λ all non-zero requires numerical solution. Analytic solutions can be found in various special cases, by setting one or other of the constants to zero.

i) Empty Models: $A = 0 \Rightarrow \dot{R} = c\sqrt{\frac{\Lambda R^2}{3} - K}$

$$\frac{dR}{\sqrt{\frac{\Lambda R^2}{3} - K}} = cdt \qquad \Rightarrow \qquad \int \frac{dR}{\sqrt{\frac{\Lambda R^2}{3} - K}} = ct$$

a) $\Lambda = 0, K = 0 \Rightarrow R = const$

b) $\Lambda = 0, K = -1 \Rightarrow R = ct,$ **Milne-universe** $(R(0) = 0)$

c) $\Lambda > 0, K = 0 \Rightarrow R(t) = R(0)\exp\left(\sqrt{\frac{\Lambda}{3}}ct\right)$

d) $\Lambda > 0, K = 1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}ct\right)$ **de Sitter space** $(R(0) = \sqrt{\frac{3}{\Lambda}})$

e) $\Lambda > 0, K = -1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}ct\right)$

f) $\Lambda < 0, K = -1 \Rightarrow R(t) = \sqrt{\frac{3}{\Lambda}}\sin\left(\sqrt{\frac{\Lambda}{3}}ct\right)$ **Oscillating universe**

ii) $A \neq 0, \Lambda = 0$

$$\dot{R} = \sqrt{\frac{A}{R} - c^2 K} \qquad \Rightarrow \qquad dt = \frac{dR}{\sqrt{\frac{A}{R} - c^2 K}}$$

a) $K = 0$, $dt = \sqrt{\frac{R}{A}}dR \Rightarrow t = \frac{2}{3\sqrt{A}}\left(R(t)^{3/2} - R(0)^{3/2}\right)$;
with initial condition $R(0) = 0$ (universe "started" at $t = 0$ with zero size)

$$t = \frac{2}{3\sqrt{A}}R(t)^{3/2}$$
$$R(t) = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}.$$

b) $K = 1$, $\dot{R} = \sqrt{\frac{A}{R} - c^2}$; solution in parametric form:

$$R(\psi) = \frac{A}{2c^2}(1 - \cos\psi)$$
$$t(\psi) = \frac{A}{2c^3}(\psi - \sin\psi) \qquad \text{(Cycloid)}$$

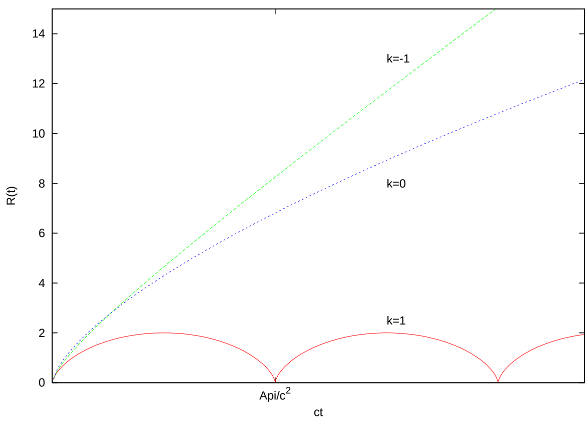
Check:

$$\dot{R} = \frac{dR}{d\psi}\frac{d\psi}{dt} = \frac{\frac{A}{2c^2}\sin\psi}{\frac{A}{2c^3}(1 - \cos\psi)} = \frac{c\sin\psi}{1 - \cos\psi}$$
$$= \frac{c\sqrt{1 - \left(\frac{2c^2}{A}R - 1\right)^2}}{\frac{2c^2R}{A}} = \frac{A}{2cR}\sqrt{\frac{4c^2R}{A} - \frac{4c^4}{A^2}R^2}$$
$$= \sqrt{\frac{A}{R} - c^2}$$

c) $K = -1$, $\dot{R} = \sqrt{\frac{A}{R} + c^2}$, a parametric solution is

$$R(\psi) = \frac{A}{2c^2}(\cosh\psi - 1)$$
$$t(\psi) = \frac{A}{2c^3}(\sinh\psi - \psi).$$

5



Models with $\Lambda = 0$ and $\rho_{Mat} \neq 0$. $R(t)$ is plotted against ct.

iii) $K = 0$ (favoured observationally).
In this case the Friemann equation reduces to

$$\dot{R}^2 = \frac{A}{R} + \frac{\Lambda c^2 R^2}{3}$$

$$\Rightarrow \quad \dot{R}^2 R = A + \frac{\Lambda c^2 R^3}{3} \quad \Rightarrow \quad 2\dot{R}\ddot{R}R + \dot{R}^3 = \Lambda c^2 R^2 \dot{R}$$

$$\Rightarrow \quad \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{2R^2} = \frac{\Lambda c^2}{2} \quad \Rightarrow \quad \frac{1}{R^{3/2}}\frac{d}{dt}\left(\dot{R}R^{1/2}\right) = \frac{\Lambda c^2}{2}$$

$$\Rightarrow \quad \frac{1}{R^{3/2}}\frac{d}{dt}\left(\frac{2}{3}\frac{d}{dt}\left(R^{3/2}\right)\right) = \frac{\Lambda c^2}{2} \quad \Rightarrow \quad \frac{d^2}{dt^2}\left(R^{3/2}\right) = \frac{3\Lambda c^2}{4}R^{3/2}$$

a) $\Lambda = 0$

$$\frac{d^2}{dt^2}\left(R^{3/2}\right) = 0 \Rightarrow R^{3/2} = a + bt \Rightarrow \frac{3}{2}\dot{R}R^{1/2} = b$$

$\frac{\dot{R}}{R} = \frac{2}{3}\frac{b}{R^{3/2}} = \frac{2}{3}\frac{b}{a+bt}$ and $\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3}$. Therefore

$$\frac{4}{9}\frac{b^2}{(a+bt)^2} = \frac{A}{(a+bt)^2}$$
$$b = \frac{3}{2}\sqrt{A}$$

Take $R(0) = 0 \Rightarrow a = 0$,

$$R^{3/2}(t) = \frac{3}{2}\sqrt{A}t$$

as before.

b) $\Lambda > 0, \Rightarrow R^{3/2} = a' e^{\sqrt{\frac{3\Lambda}{4}}ct} + b' e^{-\sqrt{\frac{3\Lambda}{4}}ct}$ or:

$$R^{3/2} = a\cosh\left(\sqrt{\frac{3\Lambda}{4}}ct\right) + b\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)$$

Taking the initial condition $R(0) = 0 \Rightarrow a = 0$

$$\Rightarrow \qquad R = b^{2/3}\left\{\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)\right\}^{2/3}$$

so

$$\frac{\dot{R}}{R} = \frac{2}{3}\sqrt{\frac{3\Lambda c^2}{4}}\frac{\cosh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)} \quad \Rightarrow \quad \left(\frac{\dot{R}}{R}\right)^2 = \frac{4}{9}\frac{3\Lambda c^2}{4}\frac{\cosh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}$$

$$\text{and} \qquad \left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3} + \frac{\Lambda c^2}{3}$$

$$\Rightarrow \qquad \frac{\Lambda c^2}{3}\frac{\cosh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)}{\sinh^2\left(\sqrt{\frac{3\Lambda}{4}}ct\right)} = \frac{A}{b^2\left\{\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)\right\}^2} + \frac{\Lambda c^2}{3}$$

$$\Rightarrow \qquad \frac{\Lambda c^2}{3}\frac{1+\sinh^2 X - \sinh^2 X}{\sinh^2 X} = \frac{A}{b^2\sinh^2 X}, \qquad X = \sqrt{\frac{3\Lambda}{4}}ct$$

$$\Rightarrow \qquad \frac{\Lambda c^2}{3} = \frac{A}{b^2} \qquad \Rightarrow \qquad b^2 = \frac{3A}{\Lambda c^2} \qquad \Rightarrow \qquad b = \sqrt{\frac{3A}{\Lambda}}\frac{1}{c}.$$

Giving the solution

$$R(t) = \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \left\{\sinh\left(\sqrt{\frac{3\Lambda}{4}}ct\right)\right\}^{2/3}$$

For small t and very large t we get the limits:

$$R(t) = \begin{cases} \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \left(\sqrt{\frac{3\Lambda}{4}}ct\right)^{2/3} = \left(\frac{9A}{4}\right)^{1/3} t^{2/3}, & t \to 0 \\ \left(\frac{3A}{\Lambda c^2}\right)^{1/3} \frac{1}{2^{2/3}} \exp\left(\frac{2}{3}\sqrt{\frac{3\Lambda}{4}}ct\right) = \left(\frac{3A}{4\Lambda c^2}\right)^{1/3} \exp\left(\sqrt{\frac{\Lambda}{3}}ct\right), & \text{late } t. \end{cases}$$

The real Universe is believed to be described by a solution of this form at the present day, intermediate between these two limits.

c) $\Lambda < 0$, $R(t) = \left(\frac{3A}{c^2|\Lambda|}\right)^{1/3} \sin^{2/3}\left(\sqrt{\frac{3|\Lambda|}{4}}ct\right)$ with the same boundary conditions as a) and b) above.

3.5 Microwave background

The universe is "glowing" at $T = 2.728K$ ($\Rightarrow \lambda = 2mm$)

i) Ordinary matter: In sphere of radius R are N galaxies, each mass m_G. The sphere contains Mass $M = Nm_G = \frac{4\pi}{3}R^3\rho_{Mat}$, $\Rightarrow \rho_{Mat} = \frac{3Nm_G}{4\pi}\frac{1}{R^3} = \frac{3A}{8\pi G}\frac{1}{R^3}$

$$\rho_{Mat} \propto \frac{1}{R^3}.$$

ii) Thermal Radiation, energy density ϵ. A sphere of radius R contains energy $E = \frac{4\pi}{3}R^3\epsilon$, but $E \propto \nu$ (for thermal radiation $k_BT \approx h\nu$). As R increases, wavelength λ stretches, $\lambda = \frac{c}{\nu} \Rightarrow \nu$ decreases $\Rightarrow \nu \propto \frac{1}{R}$, $\Rightarrow E \propto \frac{1}{R}$, $\Rightarrow R^4\epsilon = const, \epsilon \propto \frac{1}{R^4}$. Let $\rho_{Rad} = \frac{\epsilon}{c^2} = \frac{3B}{8\pi G}\frac{1}{R^4}$ with $B = const$ be the mass density equivalent to ϵ. Then

$$\rho_{Rad} \propto \frac{1}{R^4}.$$

Radiation mass density contributes to the Friedmann equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3} + \frac{B}{R^4} + \frac{\Lambda c^2}{3} - \frac{c^2K}{R^2} \tag{3.11}$$

At the present day: $K = 0, \rho_{Mat} \approx 10^{-27}kgm^{-3}, R_0 = 10^{26}m$,

$$\Rightarrow A = \frac{8\pi G}{3} \times 10^{-27} \times (10^{26})^3 \approx 6 \times 10^{41}m^3s^{-2}$$

For thermal radiation $\epsilon = \frac{\sigma_{SB}T^4}{4}$, $\Rightarrow \rho_{Rad} = 1.2 \times 10^{-31} kgm^{-3} \;\; \Rightarrow$

$$B = \frac{8\pi G}{3} \times (1.2 \times 10^{-31}) \times (10^{26})^4 \approx 6 \times 10^{63} m^4 s^{-2}$$

At the present day:

$$\frac{\rho_{Rad}}{\rho_{Mat}} = 2.8 \times 10^{-4}.$$

At an earlier time:

$$\frac{\rho_{Rad}}{\rho_{Mat}} = \frac{B}{AR(t)} = \frac{3 \times 10^{22} m}{R(t)}$$

$\Rightarrow \;\; \rho_{Rad} \geq \rho_{Mat}$ for $R(t) < 3 \times 10^{22} m$.
For matter dominated expansion (ignoring Λ),

$$R(t) = \left(\frac{9A}{3}\right)^{1/3} t^{2/3} \approx 10^{14} \times (t^{2/3}) m,$$

where t is in seconds. $R(t) = 3 \times 10^{22} m$ when $t^{2/3} = 10^8 s^{2/3} \Rightarrow t = 10^{12} s \approx 30,000 yr$.
Compare this to the current age of the Universe $t_0 = 13.7 \times 10^9 yr$.
For $t < 30,000 yr$ the energy density in radiation dominates over that in matter, so ignore A and the Friedmann equation becomes

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{B}{R^4} + \frac{\Lambda c^2}{3}.$$

But we also have $\frac{B}{R^4} \gg \frac{\Lambda c^2}{3}$ also so in the early universe ($t < 10^4 yr$) a very good approximation is

$$\dot{R}^2 = \frac{B}{R^2}$$

$\Rightarrow \;\; \dot{R}R = \sqrt{B} \;\; \Rightarrow \frac{1}{2}\frac{d}{dt}(R^2) = \sqrt{B} \;\; \Rightarrow R^2 = 2\sqrt{B}t + a$. With initial condition $R(0) = 0$ we get

$$R(t) = (2\sqrt{B})^{1/2} t^{1/2}.$$

Summary

$t > t_0 = 10^{10} yr$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda c^2}{3}$	$R(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}} ct\right)$	
$30,000 yr < t < t_0$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{A}{R^3}$	$R(t) \propto t^{2/3}$	matter dominated
$t < 30,000 yr$	$\left(\frac{\dot{R}}{R}\right)^2 = \frac{B}{R^4}$	$R(t) \propto t^{1/2}$	radiation dominated

$\frac{K}{R^2}$ never was and never will be significant.
When the universe was younger, it was hotter

$$T \propto \nu \propto \frac{1}{R}.$$

At the present day: $R_0 = 10^{26}m, T = 3K$, at $t = 30,000yr$, $\frac{R_0}{R} = 10^4 \Rightarrow T = 3 \times 10^4 K$. All the matter (mostly Hydrogen and Helium) was ionised, when $T > 4000K$. This occurred when

$$\frac{R_0}{R} \approx 10^3 \Rightarrow \frac{t_0}{t} = 10^{9/2} = 3 \times 10^4 \Rightarrow t = \frac{1.4 \times 10^{10} yr}{3 \times 10^4} = 500,000yr$$

(a more accurate figure is $t_s = 370,000\ yrs$). Before this time the matter in the Universe was an ionised plasma, afterwards it is mostly neutral Hydrogen unless it gets re-ionised due to the heat from stars.

As $R \propto t^{1/2}$ for small t we get $\dot{R} \xrightarrow{t \to 0} \infty$. We cannot trust the Friedmann equation back to $t = 0$. In practice only go back to some time t_i (a fraction of a second) where the temperature and energies are still understood and replace our ignorance of $t < t_i$ with initial conditions $R(t_i)$.

3.6 The Horizon Problem and Inflation

Microwave photons from two points 180° apart in the sky come from the "surface of last scattering" when neutral hydrogen was formed at $t_s \approx 370,000yr$ when $T \approx 4000K$. So two diametrically opposite points in the sky appear to be the same temperature (to within a few parts in 10^6), so presumably they were in thermal contact at some point in the past. But this is inconsistent with our model.

The radius R_s of the surface of last scattering can be calculated. Let our own Galaxy sit at $r = 0$ and we observe photons coming in from the surface of last scattering at a co-ordinate distance r_s. In a short time interval dt at an intermediate time t, a photon travels a physical distance $-R(t)dr = cdt \Rightarrow dr = -c\frac{dt}{R(t)}$ ($dr < 0$ because the photon is travelling inwards, towards us)

$$\int_{r_s}^{0} dr = -c \int_{t_s}^{t_0} \frac{dt}{R(t)}.$$

The physical radius at t_s is

$$R_s = R(t_s) r_s = cR(t_s) \int_{t_s}^{t_0} \frac{dt}{R(t)}.$$

For $t_s < t < t_0$ we have $R(t) = bt^{2/3}$. Therefore

$$R_s = R(t_s)\frac{1}{b}c \int_{t_s}^{t_0} \frac{dt}{t^{2/3}} = bt_s^{2/3}\frac{c}{b}\left[3t^{1/3}\right]_{t_s}^{t_0} =$$
$$= 3ct_s^{2/3}\left(t_0^{1/3} - t_s^{1/3}\right) = 3ct_s\left[\left(\frac{t_0}{t_s}\right)^{1/3} - 1\right].$$

With $t_0 \approx 1.4 \times 10^{10}\ yr, t_s \approx 3.7 \times 10^5\ yr$ we get

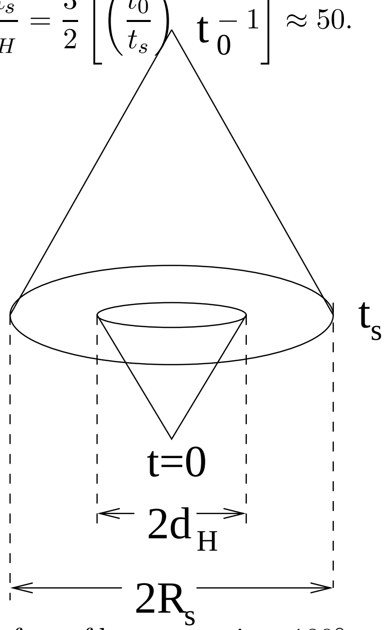
$$R_s \approx c \times 10^5 \times 30\ yr = (3 \times 10^8)\frac{m}{s} \times (3 \times 10^6) \times (3 \times 10^7 s) \approx 3 \times 10^{22} m.$$

Let the distance a photon could have travelled at time t_s, since $t = 0$, be d_H. For $t < t_s$ let $R(t) = b' t^n$ ($n = \frac{2}{3}$ for matter dominated and $n = \frac{1}{2}$ for radiation dominated). Now

$$d_H = cb' t_s^n \int_0^{t_s} \frac{dt}{b' t^n} = c t_s^n \left[\frac{t^{-n+1}}{-n+1}\right]_0^{t_s} = \frac{c}{1-n} t_s$$

e.g. for $n = \frac{1}{2} \Rightarrow d_H = 2ct_s$ but then

$$\frac{R_s}{d_H} = \frac{3}{2}\left[\left(\frac{t_0}{t_s}\right)^{1/3} - 1\right] \approx 50.$$



How can two points on the surface of last scattering, 180° apart in the sky and therefore a physical distance $2R_s$ apart be in thermal equilibrium with each other when $2R_s \approx 50(2d_H)$ is 50 times the distance a photon could have travelled since the beginning of the Universe? This is known as "the Horizon Problem".

3.6.1 Inflationary Universe

Possible solutions

1. Friedmann equation is wrong - it breaks down for some $t < t_s$ (must happen at some early time anyway, since cannot allow $\dot{R} \to \infty$) ($e.g$ quantum gravity).

2. Friedmann equation is correct, but change the R.H.S. In the "inflationary universe" picture it is assumed that a vary large positive cosmological constant "switched on" for a very brief period at a very early time ($t \sim 10^{-35} s$) and was large enough

to dominate the dynamics. If Λ dominates $\dot{R}^2 = \frac{\Lambda c^2 R^2}{3}$ and $R = R(0)e^{\sqrt{\frac{\Lambda}{3}}ct}$ exponential expansion for a period of time between t_1 and t_2. This solves the horizon problem if $R(t_2)/R(t_1) \approx 10^{25}$.

As a by-product of inflation we get a natural explanation of why $K \approx 0$. Non-zero K is associated with curvature of 3-dimensional space, for example positive curvature, $K > 0$, results in parallel lines converging, like great circles intersecting on the surface of a 2-dimensional sphere. The greater the radius of the sphere the smaller the curvature, for example it is difficult to detect the curvature of the Earth's surface on length-scales of a few metres. If $K \neq 0$ and is significant before inflation, when $t < 10^{-35}\ s$, then its significance is decreased by a factor of 10^{25} if $R(t)$ increases by 10^{25} and the relevance of K in the Friedmann becomes negligible for all times after the period of inflation. This could explain why attempts to measure K today give a null result.

3.7 The first 3 minutes (Weinberg)

Looking back at $t < 30,000yr = 10^{12}s$ when $T = 10^4 K$, $R \propto \frac{1}{T} \propto t^{1/2}$. $k_B T$ is an energy, fact $k_B T \approx \frac{1}{\sqrt{t}}$ if the energy is given in MeV ($1\ MeV \approx 10^{10}\ K$) and t in seconds.

$$\boxed{T\ (10^{10}\ K) \approx T\ (MeV) \approx \frac{1}{\sqrt{t\ (secs)}}}$$

It is remarkable how much we can deduce about the early Universe from this equation.

Time	Temperature/ Energy	
		$R \propto e^{\sqrt{\frac{\Lambda}{3}}ct} \uparrow$
$1.37 \times 10^{10} yr$	$2.7K$	t_0, present day
		$R \propto t^{2/3}$ Matter dominated $\downarrow$
$4.56 \times 10^9 yr$	$4K$	Solar System formed
$10^9 yr$	$5K$	Galaxies formed. Era of Quasars and active galactic nuclei
		Heavy elements are created in stars and supernovae
$370,000 yr$	$4000K$	Hydrogen ionises, surface of last scattering
		$R \propto t^{2/3}$ Matter dominated $\uparrow$
$10^4 yr$	$25,000K$	$R \propto t^{1/2}$ Radiation dominated $\downarrow$
		No heavy elements; Plasma, p, ^{4}He, e^-, ^{7}Li, D, γ, ν, $\bar{\nu}$
		75% p and 25% ^{4}He by mass, $\frac{1}{8}$ neutrons, $\frac{7}{8}$ protons
$100s$	$0.1MeV$	α-particles disintegrate as do D and Li, $N_n/N_p = 1/7$
	$(10^9 K)$	Era of nucleo-synthesis
$4s$	$0.5MeV$	photons in thermal background can produce e^+, e^- pairs
	$(5 \times 10^9 K)$	$p, n, e^+, e^-, \nu, \bar{\nu}, \gamma$
$1s$	$1MeV$	β-decay, $n \to p + e^- + \bar{\nu}$, starts to deplete neutrons
$10^{-2}s$	$10MeV$	$p + e^- \leftrightarrow n + \nu$ works both ways
	$(10^{11}K)$	protons and neutrons in thermal equilibrium, $N_p/N_n = 1$
$5 \times 10^{-5}s$	$150MeV$	p and n evaporate into quarks and gluons
	$(1.5 \times 10^{11}K)$	Plasma: $q, \bar{q}, e^-, e^+, \mu^-, \mu^+ \nu, \bar{\nu}, \gamma, g$
$10^{-10}s$	$100GeV$	Electromagnetism and weak nuclear force
		unify into the electro-weak force
$10^{-33}s$		Era of inflation
$10^{-38}s$	$10^{16}GeV$	Grand Unification (electromagnetism, strong and weak);
		Grand Unified Theory, Supersymmetry, Superstrings?
$10^{-43}s$	$10^{19}GeV$	Era of Quantum Gravity;
		Friedmann equation cannot be correct

Between 2 s and 100 s free neutrons can decay to protons, $n \to p + e^- + \bar{\nu}_e$. After 100 s neutrons are bound into Helium nuclei and are stable because the only energy levels available to a proton arising from neutron decay are blocked, due to the Pauli exclusion principle, by the other two protons already present in the Helium nucleus. We can estimate how many neutrons are left at 100s using the lifetime for neutron decay in free space, $n \to p + e^- + \bar{\nu}$, which is $\tau = 900\ s$. We should allow for the fact that the neutron to proton ratio N_n/N_p is not quite 1 at 2 s, due to the neutron-proton mass difference $\Delta E := (m_n - m_p)c^2 = 1.3\ MeV$. The Maxwell-Boltzmann distribution for particles of energy E in a gas at temperature T is $n(E) \sim e^{-E/k_B T}$. Using $T = 1/\sqrt{t}$, with T measured in MeV and t in seconds, gives

$$\frac{N_n}{N_p} = e^{-\Delta E \sqrt{t}} \qquad \Rightarrow \qquad N_n = N_p e^{-\Delta E \sqrt{t}}.$$

If neutrons drop out of equilibrium at a time t_e and subsequently decay to protons until they are bound into Helium nuclei at time t_{He}, then we expect the neutron-proton ratio at t_{He} to be

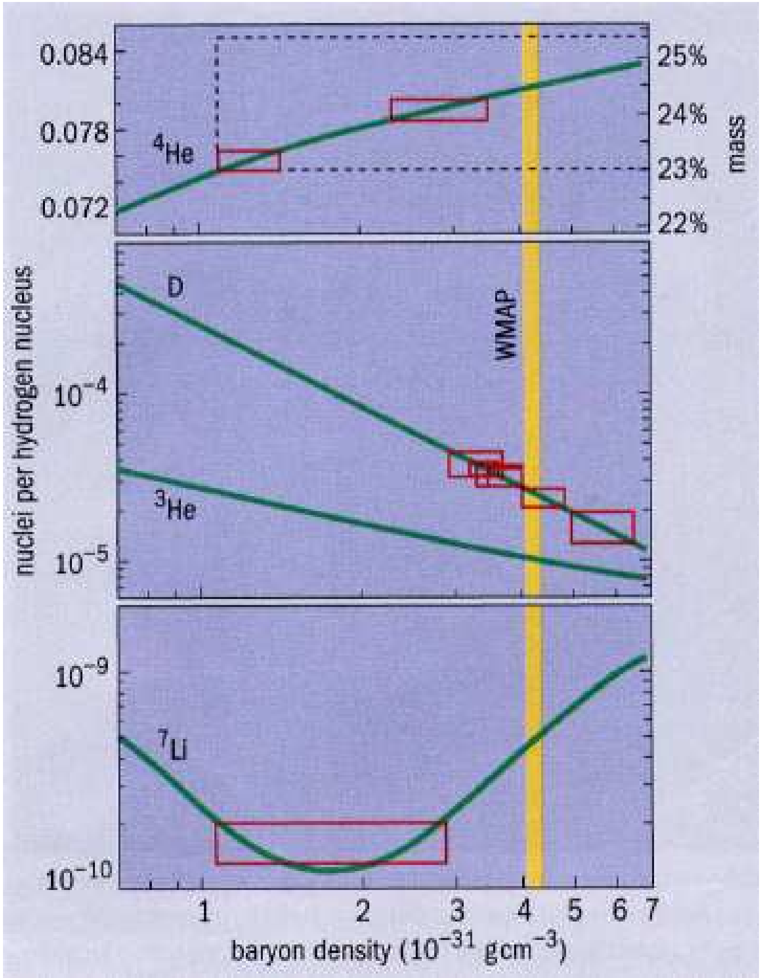
$$\frac{N_n}{N_p} = e^{-\frac{(t_{He}-t_e)}{\tau}} e^{-\Delta E \sqrt{t_e}}.$$

Using the values $\Delta E = 1.3\ MeV$, $\tau = 900\ s$, $t_e = 2\ s$ and $t_{He} = 100\ s$ gives

$$\frac{N_n}{N_p} \approx \frac{1}{7},$$

as observed.

If the rate of Helium production in the early Universe were larger, more Helium would be produced earlier and the neutrons would have less time to decay, resulting in a larger N_n/N_p ratio and hence a larger He/H ratio. If the rate of Helium production were smaller more of the Helium would be produced later and the neutrons would have more time to decay, resulting in a smaller N_n/N_p ratio and hence smaller He/H ratio. The rate for Helium production increases if the density of neutrons and protons increases and the observed ratio of primordial Hydrogen to Helium in the Universe puts a limit on the maximum allowed density of neutrons and protons consistent with observations. At the present day it cannot be more than 15% of the Dark Matter. There are similar considerations for other light elements that were produced in the Big Bang — deuterium, ^{3}He and ^{7}Li.



Abundances of primordial elements: the vertical axis is the abundance of each isotope relative to hydrogen. The red rectangles reflect the observational data — their vertical extent are the measured primordial abundances and their horizontal extent is obtained by comparing their vertical extent with the theoretical predictions as represented by the various green curves. The top curve shows the mass fraction of primordial ^{4}He relative to Hydrogen: since 1990 the abundances as measured by independent observations do not agree, probably indicating that uncertainties have been underestimated and the dotted black box may be a more accurate reflection of the observational uncertainties, giving a ^{4}He abundance lying between 0.23 and 0.25. Notice that the theoretical prediction for ^{4}He increases as the baryon density (the density of protons and neutrons) increases, as described above. The tightest constraints come from observations of inhomogeneities in the cosmic microwave background from a microwave detector called the Wilkinson Microwave Anisotropy Probe (WMAP - the vertical yellow band) indicating a value for the baryon density of $(4.1 \pm 0.1) \times 10^{-28} kgm^{-3}$, putting the density of protons plus neutrons at about 15% of the total mass density (or 4% of the critical density), implying the existence of another, unknown, type of matter. Direct observations if ^{7}Li are also somewhat lower than the WMAP value, perhaps due to the re-processing if ^{7}Li in stars not being fully understood. (Taken from Physics World, Vol. 28, No. 8, August 2007.)

At extremely high temperatures and energies the energy density in thermal radiation is so large that every photon behaves like a black-hole. Suppose a thermal photon has wavelength λ and frequency ν, so $\lambda = c/\nu$, and energy $E = h\nu$, with mass equivalent $M = E/c^2 = h\nu/c^2$. The wavelength is the same as the Schwarzschild radius when

$$\lambda \approx \frac{GM}{c^2} \qquad \Rightarrow \qquad E = h\nu = \frac{hc^3}{GM} = \frac{hc^5}{GE} \qquad \Rightarrow \qquad E = \sqrt{\frac{hc^5}{G}}.$$

It is conventional to use $\hbar$ rather than h (this is only an order of magnitude estimate) and define the *Planck Energy* as

$$\boxed{E_{Planck} = \sqrt{\frac{\hbar c^5}{G}} = 2 \times 10^9 \ J = 1.2 \times 10^{19} \ GeV}$$

and the mass equivalent is the *Planck Mass*,

$$M_{Planck} = E_{Planck}/c^2 = \sqrt{\frac{\hbar c}{G}}$$

which is about $10^{19} \ m_{proton}$ or $10^{-5} \ gm$. Dividing the Planck energy by Planck's constant gives one over the *Planck Time*,

$$t_{Planck} = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-43} \ s,$$

and multiplying this by the speed of light gives the *Planck Length*,

$$l_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} \ m.$$

The Friedmann equation is unlikely to be valid when the energy reaches the Planck energy over the Planck volume (the Planck length cubed). At these fantastic energy densities quantum effects probably require some, as yet unknown, quantum theory of gravity.

In fact the Planck energy appears in the Friedman equation naturally, at much more modest energies. Using the explicit expression for the Stefan-Boltzmann constant,

$$\sigma_{S-B} = \frac{\pi^2 k_B^4}{60\hbar^3 c^2},$$

the energy density in thermal photons is

$$\epsilon_{Rad} = \frac{4}{c}\sigma_{S-B}T^4 = \frac{\pi^2}{15c^3\hbar^3}(k_B T)^4,$$

equivalent to a mass density

$$\rho_{Rad} = \frac{\pi^2}{15c^5\hbar^3}(k_B T)^4,$$

At early times, when the universe was less than about 10,000 years old, ρ_{Rad} dominates the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho_{Rad} = \frac{8\pi^3}{45}\frac{G}{c^5\hbar^3}(k_B T)^4 = \frac{8\pi^3}{45\hbar^2}\frac{(k_B T)^4}{(E_{Planck})^2}.$$

Then the cosmic scale factor $a(t) \propto t^{1/2}$, so $H = \frac{1}{2t}$, and[1]

$$(k_B T)^4 = \frac{45}{32\pi^3}\left(\frac{\hbar E_{Planck}}{t}\right)^2.$$

This formula is valid for times as late as 10,000 years, the appearance of the Planck energy here is not a signal of quantum gravity effects, it is merely due to the fact that a classical gravitational field is being sourced by quantum matter (thermal photons).

Thanks to Tobias Hofbaur who typed the first draft of these notes from the lectures in 2005.

[1]Actually, as written, this equation is only valid for temperatures for which $k_B T << m_e c^2$, that is times later than about $4s$. For temperatures of order $2m_e c^2/k_B$ and greater electron-positron pairs can be created out of thermal energy and they contribute to ρ_{Rad}. This modifies the prefactor $\frac{8\pi^3}{45\hbar^2}$ but the general conclusion is unchanged.

Quantity	Symbol	Value
Speed of light (in vacuum)	c	299 792 458 $m\ s^{-1}$ (exact)
Newton's constant	G	$6.673 \times 10^{-11}\ kg^{-1}\ m^3\ s^{-2}$
Planck's constant	h	$6.626 \times 10^{-34}\ J\ s$
Electron charge (magnitude)	e	$1.602 \times 10^{-19}\ C$
Electric permittivity (vacuum)	$\epsilon_0 = \frac{1}{mu_0 c^2}$	$8.854 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2}$
Magnetic permeability (vacuum)	μ_0	$4\pi \times 10^{-7}\ N\ s^2\ C^{-2}$
Fine structure constant	$\alpha = \frac{e^2}{2\epsilon_0 hc}$	7.297×10^{-3}
Thompson cross-section	σ_T	$6.652 \times 10^{-29}\ m^2$
Electron mass	m_e	$9.109 \times 10^{-31}\ kg$
Proton mass	m_p	$1.673 \times 10^{-27}\ kg$
Neutron mass	m_n	$1.675 \times 10^{-27}\ kg$
Atomic mass unit (mass of ^{12}C atom /12)	$a.m.u.$	$1.661 \times 10^{-27}\ kg$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\ J\ K^{-1}$
Stefan-Boltzmann constant	σ_{SB}	$5.670 \times 10^{-8}\ J\ s^{-1}\ m^{-2}\ K^{-4}$
Avagadro's number	N_A	$6.022 \times 10^{23}\ mol^{-1}$
Earth mass	$M_\oplus$	$5.97 \times 10^{24}\ kg$
Earth radius (equatorial)	$R_\oplus$	$6.38 \times 10^3\ km$
Lunar mass	$M_☾$	$7.35 \times 10^{22}\ kg$
Lunar radius	$R_☾$	$1.74 \times 10^3\ km$
Earth-Moon distance (mean)	$d_{\oplus - ☾}$	$3.84 \times 10^5\ km$
Earth-Sun distance (mean)	$d_{\oplus - \odot}$	$1.50 \times 10^8\ km$
Solar mass	$M_\odot$	$1.99 \times 10^{30}\ kg$
Solar radius (equatorial)	$R_\odot$	$6.961 \times 10^5\ km$
Solar luminosity	$L_\odot$	$3.85 \times 10^{26}\ J\ s^{-1}$
Temperature of microwave background	T_0	$2.725 \pm 0.002°K$
Hubble constant	H_0	$72 \pm 4\ km\ s^{-1}\ Mpc^{-1}$
($H_0 = 100h\ km\ s^{-1}\ Mpc^{-1}$	h	0.72 ± 0.04)
Critical density	$\rho_c = \frac{3H_0^2}{8\pi G}$	$1.88 \times 10^{-26} h^2\ kg\ m^{-3}$
Dark energy density (Cosmological constant)	Ω_Λ	0.73 ± 0.04
Baryon density	$\Omega_B = \rho_B/\rho_{crit}$	0.044 ± 0.004
Dark matter density	$\Omega_M = \rho_M/\rho_{crit}$	0.27 ± 0.04
Total density	Ω_{tot}	1.02 ± 0.02
Age of the Universe	t_0	$13.7 \pm 0.2 \times 10^9\ yr$
Electron Volt	eV	$1.602 \times 10^{-19}\ J$
year	yr	$3.156 \times 10^7\ s$
light year	lyr	$9.461 \times 10^{15}\ m$
parsec (1pc=3.26 lyr)	pc	$3.086 \times 10^{16}\ m$