Complex Analysis

1. Complex Functions

Denote the set of all real numbers by $\mathbb{R}$. A complex number, z, is a linear combination of two real numbers, x and y,

$$z = x + iy,$$

where $i^2 = -1$. Complex numbers can be thought of as pairs of real numbers (x, y) and so can be interpreted as points in a 2-dimensional plane, called the **complex plane**, with x and y Cartesian co-ordinates in the plane. But they are more than that, because two complex numbers cam be multiplied together to give a third. The **complex conjugate** of a complex number is denote by z^* with

$$z^* = x - iy.$$

We shall denote the set of all complex numbers by $\mathbb{C}$.

A real function $f(x)$ assigns a real number $f(x)$ to every real number $x \in \mathbb{R}$. A **complex function** $f(z)$ assigns a complex number, $f(z)$, to every $z \in \mathbb{C}$. A complex function therefore has a real and an imaginary part,

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions. The function u is called the **real part** of f, and is often denoted by $\Re f$, while v is called the **imaginary part** and is denoted by $\Im f$. In general $f(z)$ can depend on z and z^* independently, e.g.

$$\begin{aligned} f(z, z^*) &= zz^* = x^2 + y^2 \qquad\qquad \text{(in this case } v = 0) \\ f(z, z^*) &= z(z^* + 1) = x^2 + y^2 + x + iy, \end{aligned}$$

but it may depend only on z (or only on z^*), e.g.

$$\begin{aligned} f(z) &= z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\ f(z) &= \mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x \mathrm{e}^{iy} = \mathrm{e}^x(\cos y + i \sin y) \\ f(z) &= \cos z = \frac{1}{2}\left(\mathrm{e}^{iz} + \mathrm{e}^{-iz}\right) = \frac{1}{2}\left(\mathrm{e}^{ix-y} + \mathrm{e}^{-ix+y}\right) \\ &= \frac{1}{2}\left(\mathrm{e}^{-y}(\cos x + i \sin x) + \mathrm{e}^{y}(\cos x - i \sin x)\right) \\ &= \cosh y \cos x - i \sinh y \sin x. \end{aligned}$$

In these last three examples $f(z)$ can be written purely as a function of z.

Sometimes it is convenient to use 2-dimensional polar co-ordinates, $x = \rho \cos \phi$ and $y = \rho \sin \phi$, so

$$z = \rho(\cos \phi + i \sin \phi) = \rho \mathrm{e}^{i\phi}.$$

In particular, for $\rho = 1$ and $\phi = \pi$, we get Euler's famous formula

$$e^{i\pi} = -1.$$

Sometimes this polar decomposition is useful in functions, for example

$$f(z) = \ln z = \ln\left(\rho \mathrm{e}^{i\phi}\right) = \ln(\rho) + i\phi.$$

In particular, for $\phi = \pi$, we have $z = -\rho$ and, with $\rho > 0$,

$$\ln(-\rho) = \ln(\rho) + i\pi$$

the logarithm of a negative number! There is a subtle ambiguity here though: since $e^{2\pi in} = 1$ for any integer n, we have $e^{i\phi} = e^{i(\phi+2\pi n)}$ and

$$\ln(z) = \ln\left(\rho e^{i(\phi+2\pi n)}\right) = \ln(\rho) + i(\phi + 2\pi n).$$

We get a different value of $\log(z)$ for every integer n. This is really just a more extreme case of the familiar ambiguity $\sqrt{x^2} = \pm|x|$ associated with real functions. Indeed

$$z^{\frac{1}{2}} = \sqrt{z} = \sqrt{\rho e^{i(\phi+2\pi n)}} = \sqrt{\rho}\, e^{\frac{i\phi}{2}} e^{i\pi n} = \pm\sqrt{\rho}\, \mathrm{e}^{\frac{i\phi}{2}},$$

where $\sqrt{\rho}$ is taken to be the positive square root and the plus sign is for n even, the minus sign for n odd. The usual real case is $\phi = 0$. The complex function $z^{\frac{1}{2}}$ is a multi-valued function, it has two different values depending on whether n is even or odd — these are called *branches* of the function. The function $f(z) = \ln(z)$ has an infinite number of branches, one for each integer n.

Many formula familiar from real analysis must be interpreted with care in complex analysis, for example

$$1^x = 1 \tag{45}$$

for any real number x.* Consider the complex expression, with n an integer,

$$1^z = (e^{2\pi in})^{(x+iy)} = e^{2\pi inx} e^{-2\pi ny}$$

and again the result is different for every n, there is an infinite number of branches. So $1^z \neq 1$ for all z, unless $n = 0$, and this has implications for raising any number to a complex power. As another example take e^z, where $z = 1 + 2\pi ik$ with k an integer. Then, since $e = e^{1+2\pi in}$,

$$e^z = \left(e^{1+2\pi in}\right)^{1+2\pi ik} = e^{(1+2\pi in)(1+2\pi ik)} = e^{1-4\pi^2 kn+2\pi i(k+n)} = e^{1-4\pi^2 kn} \tag{46}$$

* One way to prove this is to take logarithms, $\ln(1^x) = x\ln(1) = 0$ for all real x, but the only real number whose logarithm is zero is one, therefore $1^x = 1$, for all $-\infty < x < \infty$.

where, in the last equation, we have used $e^{2\pi i(k+n)} = 1$ for any integers n and k. So, for fixed k, e^z has an infinite number of branches, one for each integer n.

Now what about $(e^z)^z$? Well from (46)

$$e^z = e^{1-4\pi^2 kn},$$

so now

$$\begin{aligned}(e^z)^z &= \left(e^{1-4\pi^2 kn}.e^{2\pi i n'}\right)^z = \left(e^{1-4\pi^2 kn+2\pi i n'}\right)^{1+2\pi i k}\\ &= e^{1-4\pi^2 k(n+n')+2\pi i(n'+k-4\pi^2 k^2 n)}\\ &= e^{1-4\pi^2 k(n+n')-8\pi^3 i k^2 n} \qquad (47)\end{aligned}$$

where a *second* set of branches has been introduced, labelled by a new integer n'. We now have a double infinity of branches, labelled by the two integers n and n'. Compare this to

$$e^{z^2} = \left(e^{1+2\pi i n}\right)^{1-4\pi^2 k^2+4\pi i k}.$$

Expanding the exponent

$$\begin{aligned}e^{z^2} &= \left(e^{1+2\pi i n}\right)^{1-4\pi^2 k^2+4\pi i k}\\ &= e^{1-4\pi^2 k(2n+k)+2\pi i(2k+n'-4\pi^2 n k^2)}\\ &= e^{1-4\pi^2 k(2n+k)-8\pi^3 i n k^2)}. \qquad (48)\end{aligned}$$

Comparing (47) and (48) we see that $(e^z)^z = e^{z^2}$ if and only if

$$n + n' = 2n + k \qquad \Rightarrow \qquad n' = n + k.$$

This only makes sense, when $z = 1 + 2\pi i k$, if we are careful how we choose the branches: there is only one set of branches, labelled by n, and we *must* choose $n' = n + k$ in (47) giving

$$(e^z)^z = e^{z^2} = e^{1-4\pi^2 k(2n+k)-8\pi^3 i k^2 n}.$$

Note that the choice $n' = n + k$ is exactly what we would we have found if we had never used $e^{2\pi i(k+n)} = 1$ in the last equation of (46), but instead had kept the phase $2\pi(n + k)$ in the exponent when calculating $\left(e^z\right)^z$.

In fact it should be clear that we can iterate this to show that

$$\overbrace{\left(\left(\left(e^z\right)^z\right)^z \cdots\right)^z}^{N \text{ times}} = e^{z^N}$$

and the different branches can be consistently treated by replacing

$$e \to e^{1+2\pi i n}$$

on both sides, provided the phase is never dropped, since then

$$\overbrace{\left(\left(\left(e^z\right)^z\right)^z\cdots\right)^z}^{N\text{ times}} = \left(\left(\left(e^{1+2\pi in}\right)^{1+2\pi ik}\right)^{1+2\pi ik}\cdots\right)^{1+2\pi ik}$$
$$= e^{(1+2\pi in)\overbrace{(1+2\pi ik)\ldots(1+2\pi ik)}^{N\text{ times}}}$$
$$= e^{(1+2\pi in)(1+2\pi ik)^N} = \left(e^{1+2\pi in}\right)^{(1+2\pi ik)^N} = e^{z^N}$$

is always true, for any integer n.

Failure to choose the right branches can lead to errors, for example setting $n = n' = 0$ in (47) and (48) and assuming $\left(e^z\right)^z = e^{z^2}$ gives the equation

$$e = e^{1-4\pi^2k^2}, \tag{49}$$

which is obviously only valid for $k = 0$. There is a whole class of conundrums in the theory of complex numbers based on this kind of example,* arguments that lead to apparent paradoxes, like (49) with $k \neq 0$, which are the result of choosing the wrong branch!

2. Differentiation

When $f(z)$ is independent of z^* a natural definition of its derivative would be

$$\frac{df}{dz} := \lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z}$$

but we must be careful to check that this definition is consistent. A potential problem arises because δz can be real or imaginary, or indeed can point in any direction in the complex plane. The limit $\delta z \to 0$ only makes sense if it is independent of this direction. To determine whether or not this is the case we write $f = u + iv$, $\delta f = \delta u + i\delta v$ and $\delta z = \delta x + i\delta v$ so

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}.$$

Then, when $\delta y = 0$, we have

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta x\to 0}\left(\frac{\delta u + i\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and, when $\delta x = 0$, we have

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta y\to 0}\left(\frac{\delta u + i\delta v}{i\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

* This known as Clausen's puzzle.

Demanding that these are the same means that their real and imaginary parts must be the same, so

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.} \tag{50}$$

These conditions are not only necessary for the derivative of $f(z)$ to exist, they are also sufficient, essentially because an arbitrary direction for δz is just a linear combination of $\delta z = x$ and $\delta z = iy$.

The conditions (50) that $u(x, y)$ and $v(x, y)$ must satisfy for $\frac{df}{dz}$ to be well defined are called the **Cauchy-Riemann conditions**.

If a function $f(z)$ is differentiable in some neighbourhood of a point z_0 then it is said to be **analytic** in that neighbourhood. An alternative name, synonymous with analytic, is **holomorphic** (from the Greek $o\lambda o\sigma$ 'holos', meaning whole, and $\mu o\rho\phi\eta$ 'morphe', meaning shape). If $f(z)$ is analytic everywhere in the complex plane (except possibly for $z \to \infty$) then it is said to be an **entire** function.

Examples:

i) $$f(z) = z^2 \qquad \Rightarrow \qquad u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}.$$

The derivatives exist everywhere except for $x \to \infty$ and $y \to \infty$ so z^2 is an entire function.

ii) $$f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} \qquad \Rightarrow \qquad u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{2yx}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.$$

The derivatives exist everywhere except at the origin $x = y = 0$, so $1/z$ is analytic everywhere except at the origin.

Functions that depend on z^* are *not* analytic

iii) $$f(z^*) = z^* = x - iy \qquad \Rightarrow \qquad u(x, y) = x, \quad v(x, y) = -y$$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = -1,$$

so z^* does not satisfy the Cauchy-Riemann conditions.

iii) $$f(z, z^*) = zz^* \qquad \Rightarrow \qquad u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y, \qquad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

and again the Cauchy-Riemann conditions are not satisfied.

Analytic functions are related to the 2-dimensional Laplace equation:

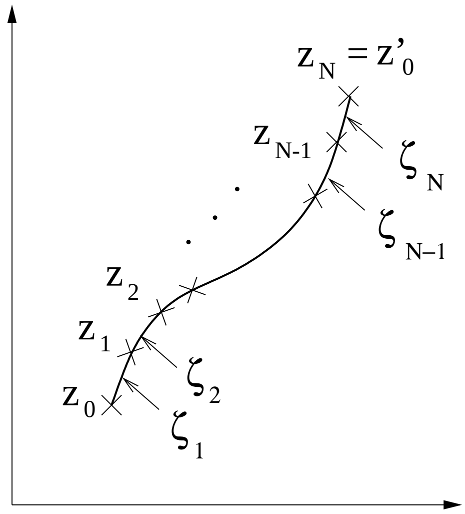
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

so both $u(x, y)$ and $v(x, y)$ satisfy the 2-dimensional Laplace equation.

3. Integration

An analytic function $f(z)$ can be integrated along a curve connecting two fixed points, z_0 and z_0', by splitting the curve up into a large number of small segments with endpoints $z_0 < z_1 < z_2 < \cdots < z_{N-1} < z_N = z_0'$ and then letting $N \to \infty$. Let $z_{j-1} < \zeta_j < z_j$ be fixed points in the j-th segment, as shown below,



then the integral can be defined as

$$\int_{z_0}^{z_0'} f(z)dz = \lim_{N \to \infty} \sum_{j=1}^{N} f(\zeta_j)(z_j - z_{j-1}),$$

and is independent of the choices for ζ_j in the limit.

Cauchy's Integral Theorem

If $f(z)$ is analytic within and on a *closed* curve C, then

$$\oint_C f(z)dz = 0.$$

The symbol $\oint$ here indicates integration around a closed loop and a closed curve C is often called a *contour* in complex integration.

Proof: first split $f(z)$ and z up into real and imaginary parts,

$$\begin{aligned}\oint_C f(z)dz &= \oint_C (u+iv)(dx+iy) \\ &= \oint_C (udx - vdy) + i\oint_C (vdx+udy).\end{aligned} \tag{51}$$

Now apply Stokes' theorem which states that, for any vector field $\underline{V} = V_x\underline{\hat{x}} + V_y\underline{\hat{y}}$ with x and y-components V_x and V_y,

$$\oint_C \big(V_x dx + V_y dy\big) = \int_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy,$$

where S is the 2-dimensional region bounded by the curve C. Applying this to the real part of (51), with $V_x = u$ and $V_y = -v$ gives

$$\oint_C (udx - vdy) = \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy,$$

which vanishes from the Cauchy-Riemann conditions (50). Similarly applying Stokes' theorem to the imaginary part of (51), with $V_x = v$ and $V_y = u$ gives

$$\oint_C (vdx + udy) = \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy,$$

which again vanishes from the Cauchy-Riemann conditions.

This shows that both the real and imaginary parts of $\oint_C f(z)dz$ vanish if $f(z)$ satisfies the Cauchy-Riemann conditions in the region S bounded by the curve C. □

An immediate consequence of Cauchy's integral theorem is that, if a function is analytic in some region containing two points z_0 and z_0', then the integral along a curve connecting z_0 and z_0' is independent of the curve chosen, provided it does not does not stray out of the region in which $f(z)$ is analytic.* This is because Cauchy's integral theorem shows that integrating from z_0 to z_0' along one curve and then integrating back from z_0' to z_0 along a different curve gives opposite answers. Combining the two integrals into a single integral along a closed curve passing through z_0 and z_0' is then zero.

Cauchy's Integral Formula

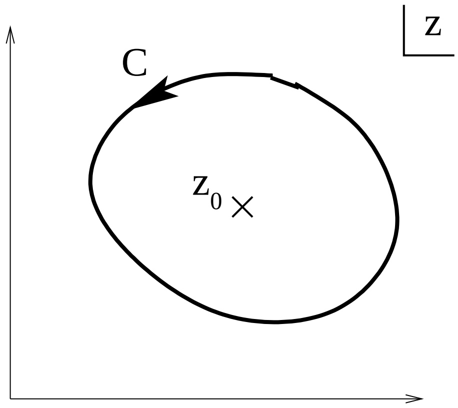
* If the real and imaginary parts $f(z)$ are thought of as potential energies for 2-dimensional problems in mechanics then this is analogous to the concept of conserved forces.

If $f(z)$ is analytic within and on a closed curve C, then

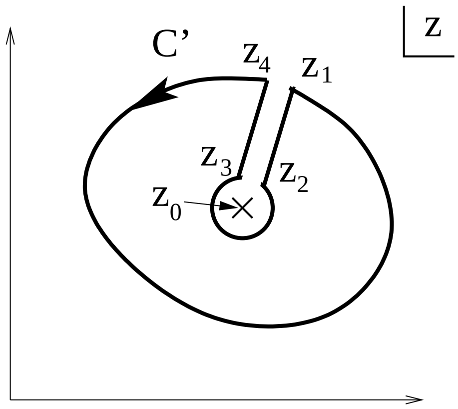
$$\oint_C \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0)$$

where z_0 is an arbitrary point inside C.

There is a convention here that the integral is performed in an anti-clockwise direction around the curve C — be careful of this: failure to adhere to this convention can introduce minus signs into some of the formulae!



Proof: although, by assumption, $f(z)$ is analytic within C, $f(z)/(z-z_0)$ is not analytic at $z = z_0$. However, if we choose a different curve C' as shown below, which excludes z_0, then $f(z)/(z-z_0)$ *is* analytic within and on C'. Hence $\oint_{C'} f(z)dz = 0$ from Cauchy's integral formula.



The curve C' can be decomposed as

$$C' = C_{12} + C_{23} + C_{34} + C_{41}$$

and then we have

$$\begin{aligned}\oint_{C'} \frac{f(z)}{(z-z_0)}dz &= \int_{C_{12}} \frac{f(z)}{(z-z_0)}dz + \int_{C_{23}} \frac{f(z)}{(z-z_0)}dz + \int_{C_{34}} \frac{f(z)}{(z-z_0)}dz + \int_{C_{41}} \frac{f(z)}{(z-z_0)}dz \\ &= 0,\end{aligned}$$

where $\int_{C_{12}} \frac{f(z)}{(z-z_0)} dz = \int_{z_1}^{z_2} \frac{f(z)}{(z-z_0)} dz$ along the segment C_{12}, etc. In the limit $z_4 \to z_1$ and $z_3 \to z_2$, the segments C_{12} and C_{34} coincide, but the integrals are performed in opposite directions, so

$$\int_{C_{12}} \frac{f(z)}{(z-z_0)} dz = -\int_{C_{34}} \frac{f(z)}{(z-z_0)} dz, \tag{52}$$

leaving

$$\oint_{C_{41}} \frac{f(z)}{(z-z_0)} dz = -\oint_{C_{23}} \frac{f(z)}{(z-z_0)} dz, \tag{53}$$

where $\lim_{z_4 \to z_1} C_{41} = C$ and C_{23} is now also a closed curve since we have set $z_2{=}z_3$. Now we choose C_{23} to be a small circle of radius ϵ centred on z_0 and parameterise the points on it by an angle ϕ, where

$$z = z_0 + \epsilon\, \mathrm{e}^{i\phi} \qquad \text{so} \qquad dz = i\epsilon\, \mathrm{e}^{i\phi} d\phi,$$

and ϕ increases in an anti-clockwise direction. We must be careful of signs because the contour C_{23} is defined above as being traversed in the clockwise direction, so this introduces a minus sign.

$$\begin{aligned}
\oint_{C_{23}} \frac{f(z)}{(z-z_0)} dz &= -\int_0^{2\pi} \frac{f(z_0 + \epsilon\, \mathrm{e}^{i\phi})}{\epsilon\, \mathrm{e}^{i\phi}} \left(i\epsilon\, e^{i\phi}\right) d\phi \\
&= -i \int_0^{2\pi} f(z_0 + \epsilon\, \mathrm{e}^{i\phi}) d\phi \\
&\underset{\epsilon \to 0}{\longrightarrow} -i \int_0^{2\pi} f(z_0) d\phi \\
&= -2\pi i f(z_0).
\end{aligned}$$

where the limit $\epsilon \to 0$ has been taken in the third step. Using this in (53), with $C_{41} = C$, now gives Cauchy's integral formula:

$$\oint_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0). \qquad \square$$

Differentiating Cauchy's integral formula with respect to z_0 gives us

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz.$$

More generally, differentiating n times we get

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \tag{54}$$

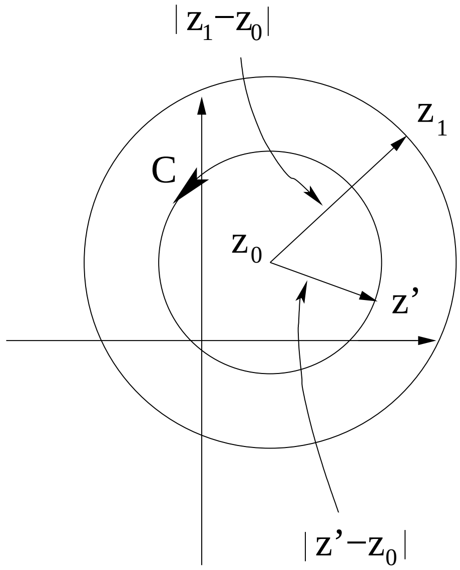
where $f^{(n)}(z_0) := \frac{d^n}{dz_0^n} f(z_0)$.

4. Laurent Expansions and Complex Analyticity

A **Laurent expansion** of a complex function is a generalisation of Taylor expansions, familiar from real analysis and we shall start by describing Taylor expansions for complex functions.

Taylor Expansions

Suppose a function $f(z)$ is analytic in some region containing a point z_0 and let z_1 be the nearest point to z_0 at which $f(z)$ is non-analytic (z_1 might be at infinity, but it could also be a finite distance from z_0, as in the picture below). Let C be a contour which encloses z_0 but does not extend as far out as z_1.



Then, from Cauchy's integral formula, for any other point z inside C

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)\left[1-\left(\frac{z-z_0}{z'-z_0}\right)\right]}. \end{aligned} \tag{55}$$

Now $\left[1-\left(\frac{z-z_0}{z'-z_0}\right)\right]^{-1}$ can be expanded in the usual way. Let $w=\frac{z-z_0}{z'-z_0}$ and $S_N=\sum_{n=0}^N w^n$. Then

$$(1-w)S_N=\sum_{n=0}^{N} w^n-\sum_{n=1}^{N+1} w^n = 1-w^{N+1}.$$

Writing $w=\rho\,\mathrm{e}^{i\phi}$ we have $\rho^{N+1}\to 0$ as $N\to\infty$, provided $\rho<1$, hence $w^{N+1}\to 0$ as $N\to\infty$ provided $|w|<1$. So we see that

$$(1-w)S_\infty = 1 \qquad \Rightarrow \qquad \frac{1}{1-w}=\sum_{n=0}^{\infty} w^n.$$

This formula should be familiar for real w but it is also valid for complex w, if $|w| < 1$. If the points z and z_0 are chosen so that $|z - z_0| < |z' - z_0| < 1$ for every point z' on the contour C, then

$$\frac{1}{1 - \left[\frac{z-z_0}{z'-z_0}\right]} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n$$

and this can be used in (55) to give

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}}.$$

Now, using (54), this can be re-expressed as

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z - z_0)^n f^{(n)}(z_0). \tag{56}$$

This is exactly the same as the usual formula for the Taylor expansion for a real function. For complex functions we see that the formula works provided $|z - z_0| < |z' - z_0|$, where z' lies on the contour used in (54). The only requirement that this contour has to satisfy is that $f(z)$ is analytic within and on C, so we may as well take C as large as we can. We can therefore take C to be a circle centred on z_0 and just inside z_1, the nearest point of non-analyticity of $f(z)$ to z_0. The Taylor expansion (56) works for any point z that is closer to z_0 than z_1.

Analytic Continuation

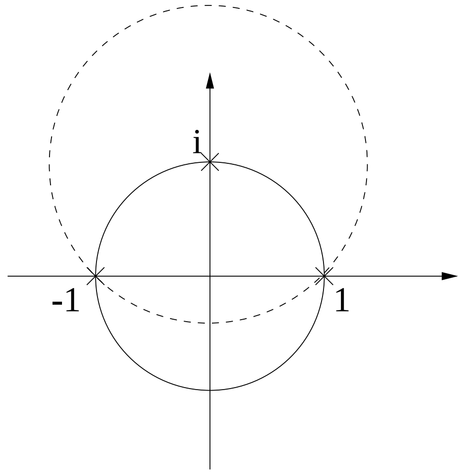
Non-analytic points are extremely important. They are obstructions to extending series expansions of $f(z)$ beyond certain limits. For example $f(z) = \frac{1}{1+z}$ is non-analytic at $z = -1$. Taylor expanding about $z_0 = 0$,

$$\frac{1}{1 + z} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n,$$

is valid provided $|z| < 1$. The unit circle centred on the origin, $|z| = 1$, is called the **circle of convergence** for this expansion. The radius of this circle, unity in this case, is called the **radius of convergence**. Strictly speaking $\frac{1}{1+z}$ and $\sum_{n=0}^{\infty}(-1)^n z^n$ are *different* functions: they co-incide for $|z| < 1$ they but are not the same for $|z| > 1$, since the former is perfectly well behaved for $|z| > 1$ while the latter is not even defined in this region as the sum diverges. We might get a different radius of convergence if we Taylor expand about a different point though. For example expanding about $z_0 = i$, we have

$$\begin{aligned} f(z) = \frac{1}{1 + z} &= \frac{1}{(1 + i)} \frac{1}{\left[1 + \left(\frac{z-i}{1+i}\right)\right]} \\ &= \frac{1}{(1 + i)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - i}{1 + i}\right)^n, \end{aligned}$$

and this expansion converges provided $\left|\frac{z-i}{1+i}\right| < 1$, i.e. provided $|z-i| < |1+i| = \sqrt{2}$. This circle of convergence has radius $\sqrt{2}$ and is centred at $z_0 = i$: it has a greater radius of convergence than the expansion about $z_0 = 0$ and the two expansions are valid in different regions. This does not mean that this expansion is any better or worse than the previous one, they are just valid in different regions as shown in the following picture. The expansion about $z_0 = 0$ is valid inside the solid circle and the expansion about $z_0 = i$ inside the dashed circle, both expansions are valid in the region where the circles overlap.



Thus the sum $\sum_{n=0}^{\infty}(-1)^n z^n$ is not defined outside $|z| < 1$, but can be continued into the crescent shaped region above the semi-circular arch between 1 and -1 by expanding about $z_0 = i$ to give $\frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{1+i}\right)^n$. In this manner the region in which the expansion is valid can be continued to different areas of the complex plane. By choosing many different points to expand about we can build up a patchwork covering the whole complex plane, excluding the point of non-analyticity $z = -1$, giving a collection of infinite sums, one for each expansion point, all of which co-incide with the function $\frac{1}{1+z}$ in the regions in which they converge. This process is called **analytic continuation** of the original expansion, $\sum_{n=0}^{\infty}(-1)^n z^n$, outside of the region $|z| < 1$.

Analytic continuation is very important in complex analysis since many functions can be defined through a series expansion, e.g.

$$\mathrm{e}^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}z^n.$$

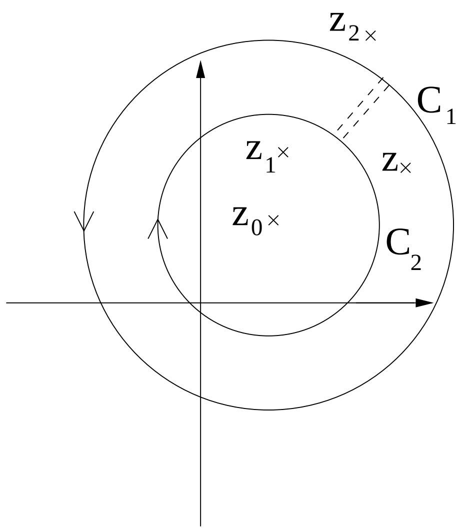
This expansion actually converges for all $|z| < \infty$, it has an infinite radius of convergence and is an example of an entire function.

Laurent Expansions

Let $f(z)$ be analytic at z_0 and let z_1 be the nearest point of non-analyticity. Suppose there are no other points of non-analyticity out as far as another point z_2. Let z be a point between z_1 and z_2, so

$$|z_1 - z_0| < |z - z_0| < |z_2 - z_0|.$$

This means that $f(z)$ is analytic in the region between the two curves C_1 and C_2 in the following figure:



We can combine C_1 and C_2 into one continuous curve, C', by cutting the annular region between them along the dotted lines indicated in the figure and including the dotted lines in the contour. The function $f(z)$ is analytic in the region enclosed by C_1, C_2 and the dotted lines, so Cauchy's integral formula (53) implies that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C'} \frac{f(z')dz'}{(z'-z)} \\ &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z'-z)} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z'-z)}. \end{aligned} \tag{57}$$

In the second equation above we have taken the limit of the two dotted line segments coinciding, so that the integrals along them exactly cancel, and we have chosen to integrate in an anti-clockwise direction along C_1, which then requires a clockwise integration around C_2 — hence the minus sign in the latter integral.

Expanded the denominators of the integrands in the same way as we did for the Taylor expansion previously we have, for C_1,

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z'-z)} &= \frac{1}{2\pi i} \oint_{C1} \frac{f(z')dz'}{(z'-z_0)\left[1-\left(\frac{z-z_0}{z'-z_0}\right)\right]} \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z'-z_0)^{n+1}}. \end{aligned} \tag{58}$$

The sum converges since, by construction, $\left|\frac{z-z_0}{z'-z_0}\right| < 1$ for z' on C_1 (see the figure above). While, for C_2,

$$\frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z'-z)} = -\frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{(z-z_0)\left[1-\left(\frac{z'-z_0}{z-z_0}\right)\right]}$$

$$= -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{(z-z_0)^{m+1}}\oint_{C_2} f(z')(z'-z_0)^m dz'$$
$$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)^{1-n}}, \quad (59)$$

where, in the last integral, the summation has been shifted by one, $n = m+1$, and the sum converges since, by construction, $\left|\frac{z'-z_0}{z-z_0}\right| < 1$ for z' on C_2. Now substitute (58) and (59) in (57) and we get

$$\begin{aligned} f(z) =& \frac{1}{2\pi i}\left\{\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} + \sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)^{-n+1}}\right\} \\ & \frac{1}{2\pi i}\left\{\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} + \sum_{n=-1}^{-\infty}(z-z_0)^{n}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)^{n+1}}\right\} \\ =& \sum_{n=-\infty}^{\infty} a_n(z_0)(z-z_0)^n, \end{aligned}$$

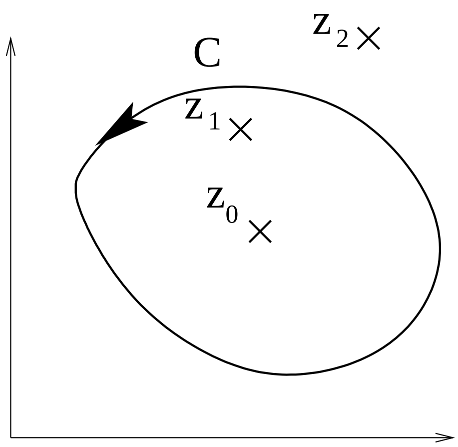
where

$$a_n(z_0) = \begin{cases} \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} & \text{for } n \geq 0 \\ \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{(z'-z_0)^{n+1}} & \text{for } n < 0. \end{cases}$$

Now observe that the integrals involved in calculating $a_n(z_0)$ are completely independent of z, they depend only on z_0, and $f(z')/(z'-z_0)^{n+1}$ is analytic in the annulus between z_2 and z_1. Due to Cauchy's integral theorem, we can distort C_1 and C_2 until they lie on top of each other, giving a single curve $C = C_1 = C_2$, without changing the value of $a_n(z_0)$ (provided neither C_1 nor C_2 passes through the points of non-analyticity, z_1 and z_2, in the process). Hence we finally arrive at

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z-z_0)^n \qquad \text{with} \qquad a_n(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}, \quad (60)$$

where C is any curve threading between the two points of non-analyticity nearest to z_0, z_1 and z_2, as shown below:



Equation (60) is called the **Laurent expansion** of $f(z)$ about z_0. Note that it differs from the Taylor expansion in that the summation contains negative powers of $(z - z_0)$, this is an inevitable consequence of the existence of the point of non-analyticity, z_1, lying between z and z_0. The summation may truncate at a finite negative power if there exists some $N > 0$ such that $a_{-n}(z_0) = 0$, $\forall n > N$.

As a final point, before going on to give an example of a Laurent expansion, note that nothing stops us taking the limit $z_1 \to z_0$ so that z_0 becomes a non-analytic point and z_2 is the next closest non-analytic point to z_0. This limit allows us to discuss expansions about non-analytic points.

Example

$$f(z) = \frac{1}{z(1-z)}$$

This function has two points of non-analyticity, $z = 0$ and $z = 1$. Now perform a Laurent expansion around the point $z_0 = 0$. The next nearest point of non-analyticity to $z = 0$ is $z = 1$, so we can use the expression for a_n in (60) with the curve C any loop around the origin which has no point greater than a unit distance from the origin, i.e. $|z'| < 1$ for all points z' on C. Equation (60) then gives

$$\begin{aligned} a_n(0) &= \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+1} z'(1-z')} \\ &= \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+2}(1-z')} \\ &= \frac{1}{2\pi i} \oint_C \frac{dz'}{(z')^{n+2}} \sum_{m=0}^{\infty} (z')^m \qquad\qquad \text{provided } |z'| < 1 \\ &= \frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint_C (z')^{m-n-2} dz'. \end{aligned}$$

By Cauchy's integral theorem the integrals are independent of the curve chosen, provided C is restricted to lie between $|z| = 1$ and $z = 0$, so we can evaluate the integrals by choosing C to be a circle of radius $a < 1$, centred on the origin. On this circle $z' = a\mathrm{e}^{i\phi'}$, so $dz' = ai\mathrm{e}^{i\phi'} d\phi'$, and

$$a_n(0) = \frac{i}{2\pi i} \sum_{m=0}^{\infty} a^{m-n-1} \int_0^{2\pi} \mathrm{e}^{i(m-n-1)\phi'} d\phi'.$$

Now

$$\begin{aligned} \int_0^{2\pi} \mathrm{e}^{i(m-n-1)\phi'} d\phi' &= \int_0^{2\pi} \cos[(m-n-1)\phi'] d\phi' + i \int_0^{2\pi} \sin[(m-n-1)\phi'] d\phi' \\ &= 2\pi \delta_{m,n+1}, \end{aligned}$$

so

$$a_n(0) = \sum_{m=0}^{\infty} a^{m-n-1}\delta_{m,n+1} = \begin{cases} 1, & n \geq -1 \\ 0, & n < -1 \end{cases} \qquad \text{independent of } a.$$

Using this in (60) gives

$$f(z) = \frac{1}{z} + 1 + z + z^2 + z^3 \cdots = \sum_{n=-1}^{\infty} z^n.$$

This example has been used to illustrate the application of the integral form of $a_n(z_0)$ in (60), but in this particular case there is a quicker way of getting the Laurent expansion. Since the function $f(z) = \frac{1}{z(1-z)}$ has two factors, $1/z$ and $1/(1-z)$, we can Taylor expand $1/(1-z)$ around $z_0 = 0$ to obtain

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

(for $|z| < 1$) and then multiply by $1/z$ to get

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} z^n = \sum_{n=-1}^{\infty} z^n$$

as before. Sometimes this can be a useful trick to obtain the Laurent expansion of a function without having to do the integrals in (60).

5. Poles and Branch Cuts

In this section we introduce a classification of the singularities (points of non-analyticity) that a function $f(z)$ might have. If $f(z)$ is non-analytic at a point z_0, but is analytic at all neighbouring points, then z_0 is called an **isolated singularity** of $f(z)$. All the non-analytic behaviour we have encountered so far (at least for functions that depend only on z and not both z and z^*) has been of this form, for example $f(z) = 1/z$ has an isolated singularity at $z = 0$. In a Laurent expansion about an isolated singularity z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n,$$

if $a_n(z_0) = 0$ for all $n < -N$ and $a_{-N}(z_0) \neq 0$, with $N > 0$, then $f(z)$ is said to have a **pole of order** N at z_0. A pole of order one is called a **simple pole**. A function which is analytic everywhere except for isolated poles of finite order is called a **meromorphic** function (from the Greek $\mu\epsilon\rho o\varsigma$ 'meros', meaning part).

Examples

i)
$$f(z) = \frac{1}{z(1-z)} = \sum_{n=-1}^{\infty} z^n$$

has $a_{-1}(0) = 1$ and $a_n(0) = 0$ for all $n < -1$, so $N = 1$ in this case, and $z = 0$ is a simple pole. The other non-analytic point, at $z = 1$, is also a simple pole.

ii)
$$f(z) = \frac{1}{z^2(1-z)} = \sum_{n=-2}^{\infty} z^n$$

has a pole of order two (sometimes called a double pole) at $z = 0$ and a simple pole at $z = 1$,

iii)
$$f(z) = \mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{|n|!} z^n.$$

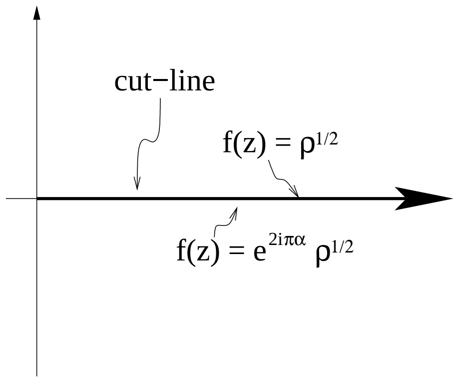
In this example the Laurent series does not terminate at any finite N, but extends all the way down to $N = -\infty$. A singularity of this form is called an **essential singularity**.

Branch Points

Consider the function
$$f(z) = z^\alpha$$
when α is not an integer. For example if $\alpha = 1/2$ then we can use the polar decomposition $z = \rho \mathrm{e}^{i\phi}$ to express $f(z)$ as
$$f(z) = \rho^{1/2} \mathrm{e}^{i\phi/2}.$$

A little thought reveals that this is actually not a very well defined function. Suppose we start at $z = 1$ and take z around the unit circle centred on $z = 0$, $z = \mathrm{e}^{i\phi}$. Writing $f = \mathrm{e}^{i\phi/2}$ as a function of ϕ we have $f(\phi = 0) = 1$ and as we go round from $\phi = 0$ to $\phi = 2\pi$ we return to $z = 1$ but we find that $f(\phi = 2\pi) = \mathrm{e}^{i\pi} = -1$. Thus $f = \pm 1$ at $z = 1$, reflecting the usual sign ambiguity in a square root. A similar ambiguity in sign is present for every value of z, except $z = 0$, so $f(z) = z^{1/2}$ really has two values for all $z \neq 0$ — it is said to be **multi-valued**. The function $z^{1/2}$ is said to have two **branches** and the point where the two branches coincide, $z = 0$, is called a **branch point**.

We can however construct a single-valued function, $f_s(z)$, from $f(z) = z^{1/2}$ at the expense of introducing a discontinuity. We first of all demand that on the positive real axis, with $\phi = 0$ and $z = \rho = x$, $f_s(x) = \sqrt{x}$, with a plus sign. We then demand that, for $0 \leq \phi < 2\pi$, the phase of $f_s(z)$ is $\phi/2$. This means that, when we go all the way round the origin to just below the positive real axis, $f_s(z)$ has changed sign, but it jumps back to the positive sign on the positive real axis. We now have a single-valued function, but it is discontinuous (and so not differentiable) across the positive real axis. Actually we could have chosen any line emanating from the origin and extending out to infinity to get a single-valued, discontinuous function in this way, but it would be a different function for different choices of line. We could even have chosen a curved line, it does not have to be straight. The line chosen in order to construct a discontinuous single-valued function associated with a continuous multi-valued function is called a **cut-line** for the function.



A cut-line for the function $f(z) = z^\alpha$, with $\alpha \notin \mathbb{Z}$.

In the example chosen above, $f(z) = z^{1/2}$, the derivative of f, $f' = \frac{1}{2}z^{-1/2}$, is singular at the branch-point $z = 0$, but this need not be the case. For example $f(z) = z^{3/2}$ also has a branch-point at $z = 0$ but its first derivative is perfectly finite there (though its second derivative is singular at $z = 0$).

6. Louiville's Theorem

A function that is everywhere finite and analytic is a constant.

Proof: By assumption $f(z)$ is everywhere finite, so $\exists K$ such that $|f(z)| < K, \forall z$. Cauchy's integral formula then implies that, for any two points z_1 and z_2,

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\oint_C \left\{\frac{1}{(z'-z_1)} - \frac{1}{(z'-z_2)}\right\} f(z')dz'$$

with C any closed curve encircling both z_1 and z_2. From this we deduce that

$$|f(z_1) - f(z_2)| < \frac{K}{2\pi}\left|\oint_C \left\{\frac{1}{(z'-z_1)} - \frac{1}{(z'-z_2)}\right\} dz'\right|.$$

We are free to chose C to be a circle of radius R centred on z_1 large enough to enclose z_2. Parameterise this circle by $z' = z_1 + R\mathrm{e}^{i\phi'}$, so that $dz' = Ri\mathrm{e}^{i\phi'}d\phi'$, and let $R > 2|z_1 - z_2|$. Then

$$\begin{aligned}\left|\oint_C \left\{\frac{1}{(z'-z_1)} - \frac{1}{(z'-z_2)}\right\} dz'\right| &= \left|\oint_C \frac{(z_1-z_2)}{(z'-z_1)(z'-z_2)}dz'\right| \\ &\le \left|\int_0^{2\pi} \frac{(z_1-z_2)}{(z'-z_2)}d\phi'\right| \qquad \text{since } |z'-z_1| = R \\ &\le \int_0^{2\pi} \frac{|z_1-z_2|}{|z'-z_2|}d\phi', \qquad \text{since } \left|\int f(z')dz'\right| \le \int |f(z')dz'|.\end{aligned}$$

Choosing $R > 2|z_2 - z_1|$ implies that $|z' - z_2| > R/2$, so

$$\begin{aligned} |f(z_1) - f(z_2)| &< \frac{K}{2\pi}\frac{|z_1 - z_2|}{(R/2)}\int_0^{2\pi} d\phi' \\ &= \frac{2K}{R}|z_2 - z_1| \\ &\underset{R\to\infty}{\longrightarrow} 0. \end{aligned}$$

Since $|f(z_1) - f(z_2)|$ cannot have any explicit dependence on R we conclude that

$$|f(z_1) - f(z_2)| = 0 \qquad \Rightarrow \qquad f(z_1) = f(z_2) \qquad \forall z_1, z_2.$$ □

7. Calculus of Residues

Cauchy's integral formula is extremely useful for calculating definite integrals. Consider the Laurent expansion of a function $f(z)$ around an isolated singular point z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z_0)(z - z_0)^n.$$

Take any contour C encircling z_0, with no other singularities inside C, and let $\tilde{z}$ be any point on C, then

$$\oint_C a_n(z_0)(z - z_0)^n dz = a_n(z_0)\left.\frac{(z - z_0)^{n+1}}{(n+1)}\right|_{\tilde{z}}^{\tilde{z}} = 0, \qquad \text{provided } n \neq -1.$$

The case $n = -1$ must be treated carefully. Since there are no singularities other than z_0 inside C, the contour can always be distorted to a circle of radius r centered on z_0 without changing the value of the integral around C. So

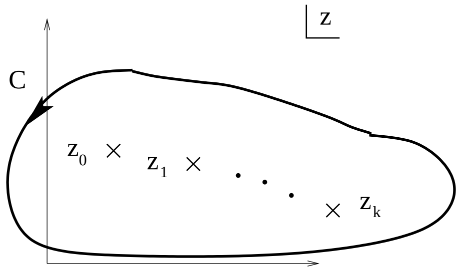
$$\oint_C a_{-1}(z_0)\frac{dz}{(z - z_0)} = a_{-1}(z_0)\int_0^{2\pi}\frac{ri e^{i\phi}d\phi}{re^{i\phi}} = 2\pi i a_{-1}(z_0),$$

where the circular contour is parameterised by $z = z_0 + re^{i\phi}$. This means that

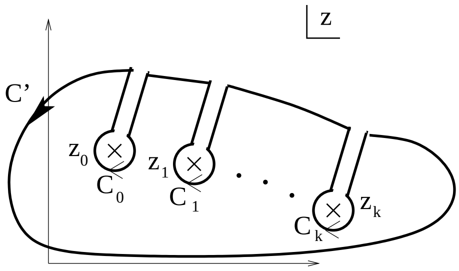
$$\frac{1}{2\pi i}\oint_C f(z)dz = a_{-1}(z_0). \tag{61}$$

This is a very powerful result, it means that we only need to know $a_{-1}(z_0)$ in order to calculate the integral — none of the other a_n's is relevant! Note that this is true for any contour C, provided only that that there are no singularities inside C other than z_0. The co-efficient $a_{-1}(z_0)$ is called the **residue** because, like the grin of the Cheshire cat, it is all that remains of $f(z)$ after the integral is performed.

If C encloses a finite set of isolated singularities, say $k+1$ of them, like this



then we can still do the integral. First deform C to C' as shown below,



Because C' does not enclose any singularities we have

$$\oint_{C'} f(z)dz = 0,$$

by Cauchy's integral theorem. Now, when the straight line segments are taken infinitesimally close to one another, the integrals along opposing edges cancel and the the perimeter becomes C, as in the proof of Cauchy's integral formula given earlier. So

$$\oint_{C'} f(z)dz = \oint_{C} f(z)dz + \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \cdots \oint_{C_k} f(z)dz = 0.$$

This can be re-written as

$$\oint_{C} f(z)dz = -\sum_{j=0}^{k} \oint_{C_j} f(z)dz. \tag{62}$$

Now each little counter C_j encloses only one singularity, so we can use the result (61), taking note of the fact that the C_j above are being traversed in a *clockwise* direction and (61) was derived by integrating around C in an anti-clockwise direction, to deduce that

$$\oint_{C_j} f(z)dz = -2\pi i a_{-1}(z_j).$$

Equation (62) now reads

$$\oint_{C} f(z)dz = 2\pi i \big\{a_{-1}(z_0) + a_{-1}(z_1) + \cdots + a_{-1}(z_k)\big\},$$

leading to

The Residue Theorem:

$$\oint_C f(z)dz = 2\pi i \sum_{j=0}^{k} a_{-1}(z_j).$$

As an example of the application of the residue theorem consider the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Treating the real line $-\infty < x < \infty$ as the horizontal axis in the complex z-plane, with $z = x + iy$, we can extend this to

$$\int_C \frac{dz}{1+z^2}$$

where C is some contour that includes the real line. For example we can take C to be a large semi-circle above the real axis, centred on $z = 0$ and radius R, with its diameter lying along the real axis. Then, on the semi-circular arch, $z = R\mathrm{e}^{i\phi}$ and $dz = Rie^{i\phi}d\phi$ with $0 \leq \phi \leq \pi$, so

$$\int_C \frac{dz}{1+z^2} = Ri\int_0^{\pi} \frac{\mathrm{e}^{i\phi}d\phi}{1+R^2\mathrm{e}^{2i\phi}} + \int_{-R}^{R} \frac{dx}{1+x^2}.$$

As $R \to \infty$ the first integral on the right-hand side vanishes so, taking the semi-circle to be of infinite radius,

$$\int_C \frac{dz}{1+z^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Now we can use the residue theorem to evaluate the left-hand side. The integrand

$$\frac{1}{1+z^2} = \frac{1}{(1+iz)}\frac{1}{(1-iz)} = \frac{1}{(z+i)}\frac{1}{(z-i)}$$

has two simple poles, one at $z = i$ and one at $z = -i$. The pole at $z = -i$ lies below the real axis and is outside the contour C, but the one at $z = i$ is inside C, and is the only point of non-analyticity inside C. Expanding about $z_0 = i$

$$\frac{1}{z+i} = \frac{1}{2i}\frac{1}{\left[1+\left(\frac{z-i}{2i}\right)\right]} = \frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

so

$$\frac{1}{1+z^2} = -\frac{1}{4}\sum_{n=-1}^{\infty}(-1)^{n+1}\left(\frac{z-i}{2i}\right)^n.$$

The residue of the simple pole at $z_0 = i$ is the $n = -1$ term:

$$a_{-1}(z_0 = i) = \frac{1}{2i}.$$

There is only one pole, and hence only one residue, within C so

$$\int_C \frac{dz}{1+z^2} = 2\pi i a_{-1}(z_0 = i) = \pi.$$

Thus we have used the residue theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

Evaluating definite integrals using the residue theorem in this way is called the **calculus of residues**.

8. Fourier Transforms

You are familiar with Fourier series for a function on an interval $-T/2 < t < T/2$.

$$f(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right).$$

where the constants A_n and B_n are determined, using orthogonality of the trigonometric functions for positive integers n and n'

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(n'\theta)d\theta = \pi\delta_{nn'} \qquad \int_{-\pi}^{\pi} \sin(n\theta)\sin(n'\theta)d\theta = \pi\delta_{nn'}$$
$$\int_{-\pi}^{\pi} \sin(n\theta)\cos(n'\theta)d\theta = 0,$$

to be

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt, \qquad B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

for positive n, while

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt.$$

In the Fourier series $\frac{n\pi}{T}$ is a frequency,

$$\omega_n = \frac{n\pi}{T},$$

and for large T the ω_n are close together for successive n, approaching a continuous variable ω as $T \to \infty$. As $T \to \infty$, the sums over n go over to Riemann integrals over angular frequency, ω, with infinitesimal frequency interval

$$d\omega = \frac{2\pi}{T}.$$

If $\int_{\infty}^{\infty} f(t)\cos(\omega_n)dt$ and $\int_{\infty}^{\infty} f(t)\sin(\omega_n)dt$ are finite A_n and B_n will vanish as $T \to \infty$ so we define

$$a(\omega_n) := \frac{T}{2\pi}A_n = \frac{1}{\pi}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt \quad \underset{T\to\infty}{\longrightarrow} \quad a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt$$
$$b(\omega_n) := \frac{T}{2\pi}B_n = \frac{1}{\pi}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt \quad \underset{T\to\infty}{\longrightarrow} \quad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt,$$

then

$$f(t) = \int_0^{\infty} \Big(a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\Big)d\omega.$$

It is conventional to define the *cosine transform* $\tilde{f}_c(\omega)$ and the *sine transform* $\tilde{f}_s(\omega)$ of $f(t)$ as

$$\tilde{f}_c(\omega) := \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos(\omega t)dt$$
$$\tilde{f}_s(\omega) := \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin(\omega t)dt.$$

These integrals certainly exist if $\int_0^{\infty} |f(t)|dt$ exists and is finite.

If $f(-t) = f(t)$ is an even function then $\tilde{f}_c(\omega) = \sqrt{\frac{\pi}{2}}a(\omega)$ and if $f(-t) = f(t)$ is an odd function then $\tilde{f}_s(\omega) = \sqrt{\frac{\pi}{2}}b(\omega)$.

Another type of integral transform that is very useful in physics when periodic phenomena are under consideration is is the *Fourier transform*,

$$\tilde{f}(\omega) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{63}$$

Specifying $\tilde{f}(\omega)$ is completely equivalent to specifying $f(t)$ because

$$f(t) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega, \tag{64}$$

as we shall now show.

Using the definition of $\tilde{f}(\omega)$ in the right hand side of (64) gives

$$\begin{aligned}\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t}d\omega =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t')e^{i\omega t'}dt'\right)e^{-i\omega t}d\omega \\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega\right)f(t')dt'.\end{aligned} \tag{65}$$

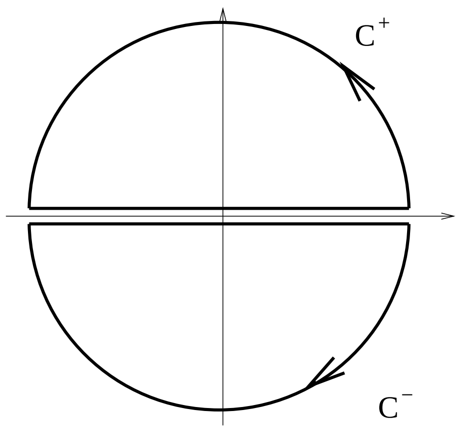
We now argue $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega$ can be interpreted as an integral representation of the Dirac delta-function. First suppose that $t \neq t'$ in the integral. The integral can be computed by using complex integration in the complex ω-plane. Consider $t < t'$ and $t > t'$ separately. When $t < t'$ $e^{i\omega(t'-t)}$ is complex analytic everywhere in the upper-half complex-ω plane, $\mathrm{Im}(\omega) > 0$, so we choose the contour C^+ below — a large semi-circle in the upper-half plane centred on the origin. As the radius of the semi-circle becomes infinite

$$\frac{1}{2\pi}\int_{C+} e^{i\omega(t'-t)}d\omega \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega = 0$$

from Cauchy's integral formula. When $t > t'$ use the contour C^- below and the same reasoning gives

$$\frac{1}{2\pi}\int_{C-} e^{i\omega(t'-t)}d\omega \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega = 0.$$

So $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega = 0$ when $t \neq t'$.



When $t = t'$ the integral is divergent. Consider

$$\int_{-T}^{T}\left(\int_{-\infty}^{\infty} e^{i\omega\tau}d\omega\right)d\tau = \int_{-\infty}^{\infty}\left[\frac{e^{i\omega\tau}}{i\omega}\right]_{-T}^{T}d\omega = 2\int_{-\infty}^{\infty}\frac{\sin(\omega T)}{\omega}d\omega = 4\int_{0}^{\infty}\frac{\sin(\omega T)}{\omega}d\omega = 2\pi,$$

which is finite $\forall T$, in particular for $T \to \infty$

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\omega\tau}d\omega\right)d\tau = 2\pi.$$

We have shown that $\int_{-\infty}^{\infty} e^{i\omega\tau}d\omega$ is zero for $\tau \neq 0$, infinite for $\tau = 0$ and its integral is finite and equal to 2π. From the defining properties of the Dirac delta-function we conclude that

$$\delta(t - t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}d\omega.$$

Using this in (65)

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{i\omega(t'-t)}d\omega\right)f(t')dt'$$
$$= \int_{-\infty}^{\infty}\delta(t'-t)f(t')dt' = f(t)$$

as claimed.

To summarise the Fourier transform and the inverse transform are*

$$\boxed{\tilde{f}(\omega) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt, \qquad f(t) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}d\omega.}$$

The Fourier, sine and cosine transforms are related. If $f(t)$ is an even function the imaginary part of $\tilde{f}(\omega)$ vanishes and

$$\tilde{f}_c(\omega) = \tilde{f}(\omega),$$

while if If $f(t)$ is an odd function the $\tilde{f}(\omega)$ is pure imaginary and

$$\tilde{f}_s(\omega) = -i\tilde{f}(\omega).$$

These integrals are examples of a class of functions called *integral transforms* where, given a function $f(t)$, we construct a new function $\tilde{f}(\omega)$ as an integral

$$\tilde{f}(\omega) := \int_{-\infty}^{\infty}f(t)K(\omega,t)dt$$

where $K(\omega, t)$ is called the *kernel* of the integral transform.

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* Note: some texts use a slightly different convention for the definition of the Fourier transform, namely

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty}f(t)e^{i\omega t}dt, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}d\omega.$$

This is more natural for physics applications, where $\nu = 2\pi\omega$ with ν a frequency in Hertz, hence $d\nu = \frac{d\omega}{2\pi}$. This convention is used in the particle physics module MP466.